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Consolidation modelling of second grade porous media by homogenization

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Thesis

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Abstract

The scope of this mater project is to investigate the macroscopic second grade modeling of a saturated porous matrix by double scale asymptotic expansion method. Strain localization is modeled by the second grade modeling which considered the second derivative displacement, and it is obtained from the virtual power formulation. The homogenization method is used to solve partial differential equations in the heterogeneous materials with a periodic structure. Each quantity (such as displacement, force, stress and water pressure) of the model is expanded (double scale) and was put them into partial differential equations. Then, the mean quantities were obtained a long with the macroscopic second grade models by the homogenization process. The modeling is equivalent to those without any boundary condition of solid or interface condition between solid and fluid inside the sample. In this report we investigated the macroscopic second grade modeling of 1D periodic medium, empty porous matrix and saturated porous matrix.

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Contents

Chapter 1

Introduction

Strain localization is an important phenomenon in geomaterials. However, classical theory of continuous media is unable to correctly model it. Different methods have been proposed to overcome the inadequacies of the classical theory of continuoum media. Second grade model considers the second derivative of the displacement field in the constitutive equation. The method is generally used to produce balance equations by virtual power Germain [6]. Chambon et al. [5] proved the one dimensional second grade model using the aforementioned method. Collin et al. [4] researched the thick-walled cylinder problem by the second grade modeling which is three dimensional.

An example of the one dimensional second grade model is presented by virtual power and it is similar to the one Chambon et al. [5]. The case study consists of a bar $(0,1)$ with zero displacement at $L(0)$ and without gravity force.

We define $N = au'$ and $M = bu''$. The internal virtual power is

$$\forall u^*, \quad p^i(u^*) = -\int_0^1 (Nu^{*\prime} + Mu^{*\prime\prime})dx \tag{1.0.1}$$

and the external virtual power is

$$\forall u^*, \quad p^e(u^*) = F_0 u^*(0) + F_1 u^*(1) + M_0 u^{*\prime}(0) + M_1 u^{*\prime}(1) \tag{1.0.2}$$

Using $u^*(0) = 0$, the virtual power formulation is

$$\int_0^1 (Nu^{*\prime} + Mu^{*\prime\prime})dx = F_1 u^*(1) + M_0 u^{*\prime}(0) + M_1 u^{*\prime}(1) \tag{1.0.3}$$

Defining $T = \frac{dM}{dx}$, the force is found to be $F_1 = N(1) - T(1)$, $M_0 = -M(0)$.

Using $u^*(0) = 0$, equation (1.0.3) is writen as

$$N(1)u^*(1) - \int_0^1 (N\frac{du^*}{dx})dx + M(1)u^{*\prime}(1) - M(0)u^{*\prime}(0) - \int_0^1 (M\frac{du^{*\prime}}{dx})dx - T(1)u^*(1) = 0 \tag{1.0.4}$$

Using the integrating by parts, we get

$$\int_0^1 \frac{dN}{dx} u^* dx + \int_0^1 \frac{dM}{dx} \frac{du^*}{dx} dx - T(1)u^*(1) = 0 \quad (1.0.5)$$

Using $u^*(0) = 0$ and the integrating by parts again, we have

$$\forall u^*, \quad \int_0^1 \frac{dN}{dx} u^* dx - \int_0^1 \frac{dT}{dx} u^* dx = 0 \quad (1.0.6)$$

Hence, the strong formulation is

$$\frac{d}{dx}(N - T) = 0 \quad (1.0.7)$$

$$T = \frac{dM}{dx} \quad (1.0.8)$$

$$N = au' \quad (1.0.9)$$

$$M = bu'' \quad (1.0.10)$$

with limit conditions $F_1 = N(1) - T(1)$, $M_0 = -M(0)$ and $u(0) = 0$.

This is the one dimensional second grade model and it is similar for three dimentional. We obtain the three dimensional second grade balance equation and boundary conditions by the virtual power and they describe the heterogeneous materials.

There are lots of examples of heterogeneous materials in the context of civil engineering, such as soil mass, rock mass and concrete etc. Heterogeneous media with a large number of heterogeneities cannot be described by considering each of the heterogeneities, which would yield to intractable boundary value problems. Generally, we use behavior of large scale (macroscopic scale) to respect the heterogeneity scale. It means using a simpler equivalent continuous medium of macroscopic behavior and macroscopic boundary conditions to research heterogeneous materials.

The homogenization method is one of equivalent methods for periodic structure. Homogenization method is also called double scale asymptotic expansions for periodic media which all coefficients and geometry are supposed to be spatially periodic. Sanchez [7] firstly used this material. It is widely used, such as in elasticity media by Sanchez [8], in filtration by Caillerie [3] and in saturated porous media by Auriault [2]. Heuristically, the method is based on the consideration of two length scales associated with the macroscopic scale (x) and microscopic scale ($y = \frac{x}{\varepsilon}$, with ε tend to zero).

The double scale asymptotic expansion method has the following steps Fig. 1.0.1):

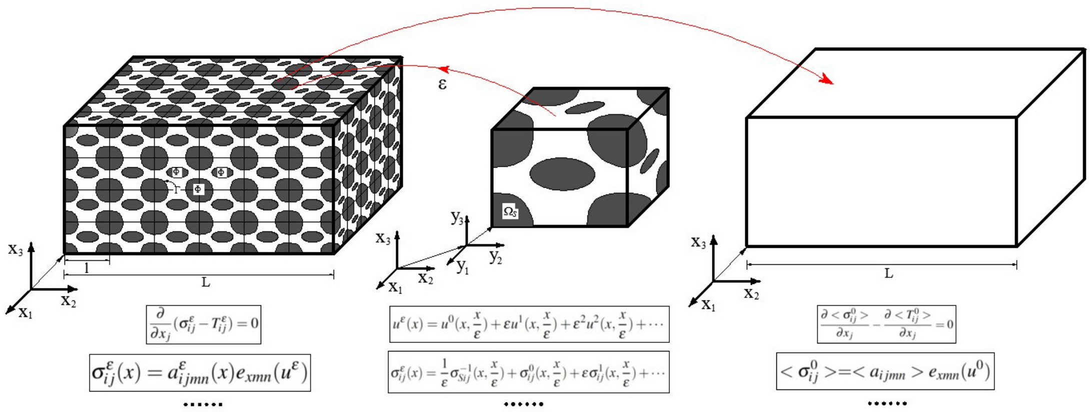

Figure 1.0.1: Method of double scale asymptotic expansions

1. We start with the second grade model of ordinary differential equations (1D problem) or partial differential equations in the heterogeneous materials with a periodic, that is macroscopic materials with variable x.

2. If a^ε_{ijmn} does not depend on x, the solid is homogeneous. On the other hand, if a^ε_{ijmn} depend on x, the solid is not homogeneous, such as solid consist of two different materials, and we define the constitutive stiffness $a^\varepsilon_{ijmn}(x) = a_{ijmn}(\frac{x}{\varepsilon}) = a_{ijmn}(y)$.

3. We look for each physical quantity, such as displacement, force, stress and water pressure (generally denoted by $\psi^\varepsilon(x)$) as double scale expansion form.

$$\psi^\varepsilon(x) = \psi^0(x, \frac{x}{\varepsilon}) + \varepsilon\psi^1(x, \frac{x}{\varepsilon}) + \varepsilon^2\psi^2(x, \frac{x}{\varepsilon}) + \cdots \tag{1.0.11}$$

The formulation above is the general form. However, some quantities may have power ε less than 0. In fact, equation (1.0.11) indicates that ψ^ε is a smooth function $\psi^0(x)$ plus a slightly high oscillating term.

4. We take the quantity of double scale expansion into partial (or ordinary) differential equations of model.

5. We have the different order partial differential equations with the boundary conditions. We can solve every physical quantity ψ^0.

6. We need to do the volume average of balance equation at ε^0 with respect to y over the period Ω, such as

$$< \frac{\partial \sigma^0_{ij}}{\partial x_j} > - < \frac{\partial T^0_{ij}}{\partial x_j} > = -\frac{1}{|\Omega|}\int_{\Omega_y}(\frac{\partial \sigma^1_{ij}}{\partial y_j} - \frac{\partial T^1_{ij}}{\partial y_j})dV \tag{1.0.12}$$

Then, we obtain the macroscpic second grade modelling by homogenzation.

This present study focuses on three macroscpic second grade models by the double scale asymptotic expansions method that are second grade modelling of one dimenional periodic media, empty porous matrix and saturated porous matrix in following chapters.

Chapter 2

Second grade modelling of one dimensional periodic medium

In this chapter, we investigate the macroscopic behaviour of second grade of one dimensional periodic medium by the double scale asymptotic expansion method. This basic problem helps us to understand how to use the asymptotic expansion approch and gives us a basic idea for empty porous matrix and saturated porous media.

2.1 One dimensional periodic medium description

The one dimensional medium under consideration has a periodic strcuture (Fig. 2.1.1), the period being small. The period Y is a segment of length L. There are more than one materials of every period and the constitutive law stiffnesses a^{ε} and b^{ε} depend on x.

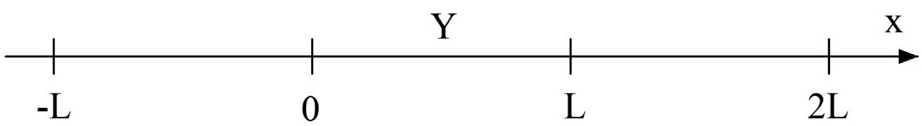

Figure 2.1.1: One dimensional periodic medium

The second grade modelling in this medium is given in the following equations in which gravity is not considered.

$$\frac{d}{dx}(N^{\varepsilon} - T^{\varepsilon}) = 0 \tag{2.1.1}$$

$$T^{\varepsilon}(x) = \frac{dM^{\varepsilon}}{dx} \tag{2.1.2}$$

$$M^{\varepsilon}(x) = b^{\varepsilon}(x)\frac{dE^{\varepsilon}}{dx} \tag{2.1.3}$$

$$N^{\varepsilon}(x) = a^{\varepsilon}(x)E^{\varepsilon} \tag{2.1.4}$$

$$E^{\varepsilon}(x) = \frac{du^{\varepsilon}}{dx} \tag{2.1.5}$$

2.2 Second grade modelling of one dimensional periodic medium

For the implementation of the asymptotic method (ε is supposed to tend to zero), the function $b^{\varepsilon}(x)$ and $a^{\varepsilon}(x)$ are defined by :

$$a^{\varepsilon}(x) = b(\frac{x}{\varepsilon}) = a(y) \tag{2.2.1}$$

$$b^{\varepsilon}(x) = b(\frac{x}{\varepsilon}) = b(y) \tag{2.2.2}$$

We look for the displacement $u^{\varepsilon}(x)$ as double scale expansion,

$$u^{\varepsilon}(x) = u^0(x,\frac{x}{\varepsilon}) + \varepsilon u^1(x,\frac{x}{\varepsilon}) + \varepsilon^2 u^2(x,\frac{x}{\varepsilon}) + \varepsilon^3 u^3(x,\frac{x}{\varepsilon}) + \cdots \tag{2.2.3}$$

The normal force $N^{\varepsilon}(x)$ is the first derivative $u^{\varepsilon}(x)$ and the force $M^{\varepsilon}(x)$ is the second grade $u^{\varepsilon}(x)$. The ε power of first term of force expansion is the same as the first term of the constitutive equation expansion. Hence, we look for $N^{\varepsilon}(x)$ and $M^{\varepsilon}(x)$

$$N^{\varepsilon}(x) = \frac{1}{\varepsilon}N^{-1}(x,\frac{x}{\varepsilon}) + N^0(x,\frac{x}{\varepsilon}) + \varepsilon N^0(x,\frac{x}{\varepsilon}) + \cdots \tag{2.2.4}$$

$$M^{\varepsilon}(x) = \frac{1}{\varepsilon^2}M^{-2}(x,\frac{x}{\varepsilon}) + \frac{1}{\varepsilon}M^{-1}(x,\frac{x}{\varepsilon}) + M^0(x,\frac{x}{\varepsilon}) + \varepsilon M^1(x,\frac{x}{\varepsilon}) + \cdots \tag{2.2.5}$$

with $y = x/\varepsilon$. u^i, N^i and M^i are y-periodic, with period Y.

When substituting the expansions (2.2.3) and (2.2.5) into equations (2.1.5) and (2.1.2), it is convenient to replace the operator $\frac{\partial}{\partial x_i}$ by $\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}$ which yields:

$$E^{\varepsilon}(x) = \frac{1}{\varepsilon}\frac{\partial u^0}{\partial y} + \frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y} + \varepsilon(\frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y}) + \varepsilon^2(\frac{\partial u^2}{\partial x} + \frac{\partial u^3}{\partial y}) + \cdots \tag{2.2.6}$$

$$T^{\varepsilon}(x) = \frac{1}{\varepsilon^3}\frac{\partial M^{-2}}{\partial y} + \frac{1}{\varepsilon^2}(\frac{\partial M^{-2}}{\partial x} + \frac{\partial M^{-1}}{\partial y}) + \frac{1}{\varepsilon}(\frac{\partial M^{-1}}{\partial x} + \frac{\partial M^{0}}{\partial y}) + \frac{\partial M^{0}}{\partial x} + \frac{\partial M^{1}}{\partial y} + \cdots \tag{2.2.7}$$

We define $T^{-3}(x,y) = \frac{\partial M^{-2}}{\partial y}$, $T^{-2}(x,y) = \frac{\partial M^{-2}}{\partial x} + \frac{\partial M^{-1}}{\partial y}$, $T^{-1}(x,y) = \frac{\partial M^{-1}}{\partial x} + \frac{\partial M^{0}}{\partial y} \cdots$ and equation (2.2.7) is written as

$$T^{\varepsilon}(x) = \frac{1}{\varepsilon^3}T^{-3}(x,y) + \frac{1}{\varepsilon^2}T^{-2}(x,y) + \frac{1}{\varepsilon}T^{-1}(x,y) + T^{0}(x,y) + \cdots \tag{2.2.8}$$

Using the expansions (2.2.8) and (2.2.4) into the balance equation (2.1.1), we get

$$\begin{aligned}
&-\frac{1}{\varepsilon^4}\frac{\partial T^{-3}}{\partial y} - \frac{1}{\varepsilon^3}(\frac{\partial T^{-3}}{\partial x} + \frac{\partial T^{-2}}{\partial y}) \\
&-\frac{1}{\varepsilon^2}(\frac{\partial T^{-2}}{\partial x} - \frac{\partial N^{-1}}{\partial y} + \frac{\partial T^{-1}}{\partial y}) + \frac{1}{\varepsilon}(\frac{\partial N^{-1}}{\partial x} - \frac{\partial T^{-1}}{\partial x} + \frac{\partial N^{0}}{\partial y} - \frac{\partial T^{0}}{\partial y}) \\
&+(\frac{\partial N^{0}}{\partial x} - \frac{\partial T^{0}}{\partial x} + \frac{\partial N^{1}}{\partial y} - \frac{\partial T^{1}}{\partial y}) + \varepsilon(\frac{\partial N^{1}}{\partial x} - \frac{\partial T^{1}}{\partial x} + \frac{\partial N^{2}}{\partial y} - \frac{\partial T^{2}}{\partial y}) + \cdots = 0
\end{aligned} \tag{2.2.9}$$

Using the expansions (2.2.6) and (2.2.4) into the constitutive equation (2.1.4), we obtain

$$\frac{1}{\varepsilon}N^{-1}(x,y) + N^{0}(x,y) + \varepsilon N^{0}(x,y) + \cdots = \frac{1}{\varepsilon}a\frac{\partial u^0}{\partial y} + a(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y}) + \varepsilon a(\frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y}) + \cdots \tag{2.2.10}$$

Using the expansions (2.2.5) and (2.2.6) into the constitutive equation (2.1.3) is writen

$$\begin{aligned}
& \frac{1}{\varepsilon^2}M^{-2}(x,y) + \frac{1}{\varepsilon}M^{-1}(x,y) + M^{0}(x,y) + \cdots \\
= \quad & \frac{1}{\varepsilon^2}b\frac{\partial^2 u^0}{\partial y^2} + \frac{1}{\varepsilon}b(2\frac{\partial^2 u^0}{\partial x \partial y} + \frac{\partial^2 u^1}{\partial y^2}) + b(\frac{\partial^2 u^0}{\partial x^2} + 2\frac{\partial^2 u^1}{\partial x \partial y} + \frac{\partial^2 u^2}{\partial y^2}) + \cdots
\end{aligned} \tag{2.2.11}$$

One of periods is Y in inside medium, for example $(0,L)$.

We obtain at the first order a periodicity boundary value problem for u^0:

$$\frac{\partial T^{-3}}{\partial y} = 0 \tag{2.2.12}$$

$$T^{-3}(x,y) = \frac{\partial M^{-2}}{\partial y} \tag{2.2.13}$$

$$M^{-2}(x,y) = b(y)\frac{\partial^2 u^0}{\partial y^2} \tag{2.2.14}$$

and the periodicity conditions

The integral in equation (2.2.12) with respect to y is

$$T^{-3}(x,y) = \alpha_1 \tag{2.2.15}$$

where α_1 does not depend on y.

Equation $\frac{\partial M^{-2}}{\partial y} = \alpha_1$ is integrable with respect to y again

$$M^{-2}(x,y) = \alpha_1 y + \beta_1 \tag{2.2.16}$$

where β_1 does not depend on y.

Y is periodic thus $M^{-2}(x,0) = M^{-2}(x,L)$. We get $\alpha_1 = 0$ and $M^{-2}(x,y) = \beta_1$. Equation $\frac{\partial^2 u^0}{\partial y^2} = \frac{\beta_1}{b(y)}$ is integrable with respect to y

$$\frac{\partial u^0}{\partial y} = \beta_1 \int_0^y \frac{1}{b(y)} dy + \gamma_1 \tag{2.2.17}$$

where γ_1 does not depend on y.

Due to fact that Y is periodic, we get $\frac{\partial u^0}{\partial y}(x,0) = \frac{\partial u^0}{\partial y}(x,L)$. Consequently, we obtain $\beta_1 = 0$ and $M^{-2}(x,y) = 0$. Equation $\frac{\partial u^0}{\partial y} = \gamma_1$ is integrable with respect to y

$$u^0(x,y) = \gamma_1 y + \delta_1 \tag{2.2.18}$$

where δ_1 does not depend on y.

Y is periodic thus $u^0(x,0) = u^0(x,L)$. Then we obtain $\gamma_1 = 0$ and $\frac{\partial u^0}{\partial y} = 0$.

Summarizing, we obtain $M^{-2}(x,y) = 0$ and $u^0 = u^0(x)$.

At the following order, we obtain a periodicity boundary value problem for u^1.

$$\frac{\partial T^{-3}}{\partial x} + \frac{\partial T^{-2}}{\partial y} = 0 \tag{2.2.19}$$

$$T^{-3}(x,y) = \frac{\partial M^{-2}}{\partial y} \qquad \textit{and} \qquad T^{-2}(x,y) = \frac{\partial M^{-2}}{\partial x} + \frac{\partial M^{-1}}{\partial y} \tag{2.2.20}$$

$$M^{-1}(x,y) = b(y)(2\frac{\partial^2 u^0}{\partial x \partial y} + \frac{\partial^2 u^1}{\partial y^2}) \tag{2.2.21}$$

and the periodicity conditions

Using $M^{-2}(x,y) = 0$ and $u^0 = u^0(x)$, we get that

$$\frac{\partial T^{-2}}{\partial y} = 0 \tag{2.2.22}$$

$$T^{-2}(x,y) = \frac{\partial M^{-1}}{\partial y} \tag{2.2.23}$$

$$M^{-1}(x,y) = b(y)\frac{\partial^2 u^1}{\partial y^2} \tag{2.2.24}$$

and the periodicity conditions

The form is same as first order. Furthermore, we obtain $M^{-1}(x,y) = 0$ and $u^1 = u^1(x)$.

At the following order, we obtain a periodicity boundary value problem for u^2.

$$\frac{\partial T^{-2}}{\partial x} - \frac{\partial N^{-1}}{\partial y} + \frac{\partial T^{-1}}{\partial y} = 0 \tag{2.2.25}$$

$$T^{-2}(x,y) = \frac{\partial M^{-2}}{\partial x} + \frac{\partial M^{-1}}{\partial y} \quad \textit{and} \quad T^{-1}(x,y) = \frac{\partial M^{-1}}{\partial x} + \frac{\partial M^{0}}{\partial y} \tag{2.2.26}$$

$$M^{0}(x,y) = b(y)(\frac{\partial^2 u^0}{\partial x^2} + 2\frac{\partial^2 u^1}{\partial x \partial y} + \frac{\partial^2 u^2}{\partial y^2}) \tag{2.2.27}$$

$$N^{-1}(x,y) = a(y)\frac{\partial u^0}{\partial y} \tag{2.2.28}$$

and the periodicity conditions

Using $M^{-2}(x,y) = 0$, $u^0 = u^0(x)$, $M^{-1}(x,y) = 0$ and $u^1 = u^1(x)$, we have that

$$\frac{\partial T^{-1}}{\partial y} = 0 \tag{2.2.29}$$

$$T^{-1}(x,y) = \frac{\partial M^{0}}{\partial y} \tag{2.2.30}$$

$$M^{0}(x,y) = b(y)(\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2}) \tag{2.2.31}$$

and the periodicity conditions

By the same way as the first order, we get $T^{-1} = 0$ and M^0 does not depend on y.

Equation (2.2.31) is written as

$$\frac{M^0(x)}{b(y)} = \frac{\partial^2 u^0}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} \tag{2.2.32}$$

Equation (2.2.32) is integrable with respect to y.

$$M^0(x) \int_0^y \frac{1}{b(y)} dy = y \frac{\partial^2 u^0}{\partial x^2} + \int_0^y \frac{\partial^2 u^2}{\partial y^2} dy \tag{2.2.33}$$

We have $\frac{\partial u^2}{\partial y}(x,0) = \frac{\partial u^2}{\partial y}(x,L)$ because Y is periodic and equation (2.2.33) is written as

$$M^0(x) = \frac{L}{\int_0^L \frac{1}{b(y)} dy} \frac{\partial^2 u^0}{\partial x^2} \tag{2.2.34}$$

At the following order, we obtain a periodicity boundary value problem for u^3.

$$\frac{\partial N^{-1}}{\partial x} - \frac{\partial T^{-1}}{\partial x} + \frac{\partial N^0}{\partial y} - \frac{\partial T^0}{\partial y} = 0 \tag{2.2.35}$$

$$T^{-1}(x,y) = \frac{\partial M^{-1}}{\partial x} + \frac{\partial M^0}{\partial y} \qquad and \qquad T^0(x,y) = \frac{\partial M^0}{\partial x} + \frac{\partial M^1}{\partial y} \tag{2.2.36}$$

$$M^1(x,y) = b(y)\left(\frac{\partial^2 u^1}{\partial x^2} + 2\frac{\partial^2 u^2}{\partial x \partial y} + \frac{\partial^2 u^3}{\partial y^2}\right) \tag{2.2.37}$$

$$N^0(x,y) = a(y)\left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y}\right) \tag{2.2.38}$$

and the periodicity conditions

Using $u^1 = u^1(x)$, we have

$$N^0(x,y) = a(y) \frac{\partial u^0}{\partial x} \tag{2.2.39}$$

Consider the balance equation at order ε^0, we obtain the following equations

$$\frac{\partial N^0}{\partial x} - \frac{\partial T^0}{\partial x} + \frac{\partial N^1}{\partial y} - \frac{\partial T^1}{\partial y} = 0 \tag{2.2.40}$$

$$T^0(x,y) = \frac{\partial M^0}{\partial x} + \frac{\partial M^1}{\partial y} \tag{2.2.41}$$

and the periodicity conditions

We average the equation (2.2.40) in periodic Y with respect to y and we obtain

$$< \frac{\partial N^0}{\partial x} > - < \frac{\partial T^0}{\partial x} > + \frac{1}{L}\int_0^L \frac{\partial N^1}{\partial y} dy - \frac{1}{L}\int_0^L \frac{\partial T^1}{\partial y} dy = 0 \tag{2.2.42}$$

Using the periodicity of Y, we have $N^1(x,0) = N^1(x,L)$ and $T^1(x,0) = T^1(x,L)$ and we have that

$$< \frac{\partial N^0}{\partial x} > - < \frac{\partial T^0}{\partial x} >= 0 \tag{2.2.43}$$

Function N^0 and T^0 are continuous functions so we can exchange order of integration and derivation.

$$< \frac{\partial N^0}{\partial x} >= \frac{1}{L}\int_0^L \frac{\partial N^0}{\partial x} dy = \frac{\partial}{\partial x}(\frac{1}{L}\int_0^L N^0 dy) = \frac{\partial < N^0 >}{\partial x} \tag{2.2.44}$$

We also have $< \frac{\partial T^0}{\partial x} >= \frac{\partial < T^0 >}{\partial x}$ and equation (2.2.43) is written

$$\frac{\partial < N^0 >}{\partial x} - \frac{\partial < T^0 >}{\partial x} = 0 \tag{2.2.45}$$

We average equation (2.2.41) in periodic Y with respect to y and we have

$$< T^0(x,y) >=< \frac{\partial M^0}{\partial x} > + \frac{1}{L}\int_0^L \frac{\partial M^1}{\partial y} dy \tag{2.2.46}$$

Using $M^1(x,0) = M^1(x,L)$ and exchanging ortder of integration and derivation, we get

$$< T^0(x) >=< \frac{\partial M^0}{\partial x} >= \frac{\partial < M^0 >}{\partial x} \tag{2.2.47}$$

We obtain macroscpic second grade modelling of 1D periodic medium by homogenization

$$\frac{\partial < N^0(x) >}{\partial x} - \frac{\partial < T^0(x) >}{\partial x} = 0 \tag{2.2.48}$$

$$< T^0(x) >= \frac{\partial < M^0(x) >}{\partial x} \tag{2.2.49}$$

$$< M^0(x) >=< \frac{L}{\int_0^L \frac{1}{b(y)} dy} > \frac{\partial^2 u^0}{\partial x^2} \tag{2.2.50}$$

$$< N^0(x) >=< a(y) > \frac{\partial u^0}{\partial x} \tag{2.2.51}$$

Chapter 3

Second grade modelling of empty porous matrix

In this chapter, we investigate the macroscopic behaviour of an empty porous matrix, which will appear as a basic problem for the saturated case in the next chapter. The present study following Auriault [2] justifies the macroscopic modelling of empty porous matrix and saturated porous matrix by homogenization without second grade modelling.

3.1 Empty porous matrix description

We investigate the behavior of a periodic Galilean porous matrix with empty pores (no stress on Γ) (Fig. 3.1.1).

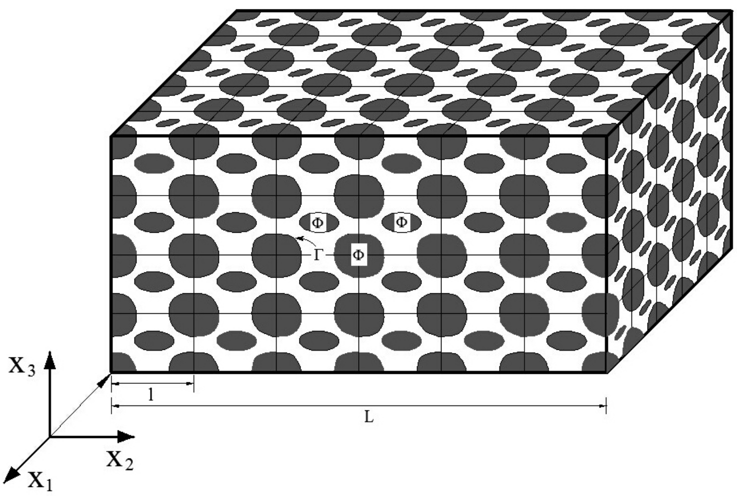

Figure 3.1.1: Periodic empty porous matrix

There are solid Φ, boundary Γ and small period Ω. a_{ijmn} and b_{ijkhmn} are constants because there is the solid consist of only one material. The second modelling of stress $\sigma^{\varepsilon}_{ij}(x)$, second grade stress $\chi^{\varepsilon}_{ijk}(x)$ and displacement $u^{\varepsilon}(x)$ of the matrix verify the following equations where gravity is not considered:

in solid Φ

$$\frac{\partial}{\partial x_j}(\sigma^{\varepsilon}_{ij} - T^{\varepsilon}_{ij}) = 0 \tag{3.1.1}$$

$$\sigma^{\varepsilon}_{ij}(x) = a_{ijmn} e_{xmn}(u^{\varepsilon}) \tag{3.1.2}$$

$$T^{\varepsilon}_{ij}(x) = \frac{\partial \chi^{\varepsilon}_{ijk}}{\partial x_k} \tag{3.1.3}$$

$$\chi^{\varepsilon}_{ijk}(x) = b_{ijkhmn} \frac{\partial e_{xmn}(u^{\varepsilon})}{\partial x_h} \tag{3.1.4}$$

$$e_{xmn}(u^{\varepsilon}) = \frac{1}{2}\left(\frac{\partial u^{\varepsilon}_m}{\partial x_n} + \frac{\partial u^{\varepsilon}_n}{\partial x_m}\right) \tag{3.1.5}$$

on boundary Γ

$$\chi^{\varepsilon}_{ijk} n_k = 0 \qquad on \quad \Gamma \tag{3.1.6}$$

$$(\sigma^{\varepsilon}_{ij} - T^{\varepsilon}_{ij}) N_j = 0 \qquad on \quad \Gamma \tag{3.1.7}$$

and boundary conditions outside the sample (unspecified here)

3.2 Second grade modelling of empty porous matrix

We look for $u^{\varepsilon}(x)$ as double scale expansion, of the form:

$$u^{\varepsilon}(x) = u^0(x, \frac{x}{\varepsilon}) + \varepsilon u^1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u^2(x, \frac{x}{\varepsilon}) + \cdots \tag{3.2.1}$$

and the form of stress $\sigma^{\varepsilon}_{ij}(x)$ and second grade stress $\chi^{\varepsilon}_{ijk}(x)$ for the same reason as the 1D periodic media,

$$\sigma^{\varepsilon}_{ij}(x) = \frac{1}{\varepsilon}\sigma^{-1}_{ij}(x, \frac{x}{\varepsilon}) + \sigma^{0}_{ij}(x, \frac{x}{\varepsilon}) + \varepsilon \sigma^{1}_{ij}(x, \frac{x}{\varepsilon}) + \varepsilon^2 \sigma^{2}_{ij}(x, \frac{x}{\varepsilon}) + \cdots \tag{3.2.2}$$

$$\chi^{\varepsilon}_{ijk}(x) = \frac{1}{\varepsilon^2}\chi^{-2}_{ijk}(x, \frac{x}{\varepsilon}) + \frac{1}{\varepsilon}\chi^{-1}_{ijk}(x, \frac{x}{\varepsilon}) + \chi^{0}_{ijk}(x, \frac{x}{\varepsilon}) + \varepsilon \chi^{1}_{ijk}(x, \frac{x}{\varepsilon}) + \cdots \tag{3.2.3}$$

with $y = x/\varepsilon$, and where σ^i, χ^i_{ijk} and u^i are $y-$periodic with period Ω.

When substituting the expansions (3.2.1) and (3.2.3) into the equations (3.1.5) and (3.1.3), we have

$$e_{xmn}(u^\varepsilon) = \frac{1}{\varepsilon} e_{ymn}(u^0) + e_{xmn}(u^0) + e_{ymn}(u^1) + \varepsilon(e_{xmn}(u^1) + e_{ymn}(u^2)) + \cdots \tag{3.2.4}$$

$$T^\varepsilon_{ij}(x) = \frac{1}{\varepsilon^3}\frac{\partial \chi^{-2}_{ijk}}{\partial y_k} + \frac{1}{\varepsilon^2}(\frac{\partial \chi^{-2}_{ijk}}{\partial x_k} + \frac{\partial \chi^{-1}_{ijk}}{\partial y_k}) + \frac{1}{\varepsilon}(\frac{\partial \chi^{-1}_{ijk}}{\partial x_k} + \frac{\partial \chi^{0}_{ijk}}{\partial y_k}) + \cdots \tag{3.2.5}$$

We define

$$T^\varepsilon_{ij}(x) = \frac{1}{\varepsilon^3} T^{-3}_{ij} + \frac{1}{\varepsilon^2} T^{-2}_{ij} + \frac{1}{\varepsilon} T^{-1}_{ij} + \cdots \tag{3.2.6}$$

where $T^{-3}_{ij} = \frac{\partial \chi^{-2}_{ijk}}{\partial y_k}$, $T^{-2}_{ij} = \frac{\partial \chi^{-2}_{ijk}}{\partial x_k} + \frac{\partial \chi^{-1}_{ijk}}{\partial y_k}$, $T^{-1}_{ij} = \frac{\partial \chi^{-1}_{ijk}}{\partial x_k} + \frac{\partial \chi^{0}_{ijk}}{\partial y_k}$...

Using the expansion (3.2.6) into the banlance equation (3.1.1), we get

$$\begin{aligned} &-\frac{1}{\varepsilon^4}\frac{\partial T^{-3}_{ij}}{\partial y_j} - \frac{1}{\varepsilon^3}(\frac{\partial T^{-3}_{ij}}{\partial x_j} + \frac{\partial T^{-2}_{ij}}{\partial y_j}) - \frac{1}{\varepsilon^2}(\frac{\partial T^{-2}_{ij}}{\partial x_j} - \frac{\partial \sigma^{-1}_{ij}}{\partial y_j} + \frac{\partial T^{-1}_{ij}}{\partial y_j}) \\ &+\frac{1}{\varepsilon}(\frac{\partial \sigma^{-1}_{ij}}{\partial x_j} - \frac{\partial T^{-1}_{ij}}{\partial x_j} + \frac{\partial \sigma^{0}_{ij}}{\partial y_j} - \frac{\partial T^{0}_{ij}}{\partial y_j}) + (\frac{\partial \sigma^{0}_{ij}}{\partial x_j} - \frac{\partial T^{0}_{ij}}{\partial x_j} + \frac{\partial \sigma^{1}_{ij}}{\partial y_j} - \frac{\partial T^{1}_{ij}}{\partial y_j}) + \cdots = 0 \end{aligned} \tag{3.2.7}$$

By using the expansions (3.2.4) and (3.2.2) into the constitutive equation (3.1.2), we have

$$\begin{aligned} & \frac{1}{\varepsilon}\sigma^{-1}_{ij}(x,y) + \sigma^0_{ij}(x,y) + \varepsilon\sigma^1_{ij}(x,y) + \cdots \\ = & \frac{1}{\varepsilon} a_{ijmn} e_{ymn}(u^0) + a_{ijmn}(e_{xmn}(u^0) + e_{ymn}(u^1)) + \varepsilon a_{ijmn}(e_{xmn}(u^1) + e_{ymn}(u^2)) + \cdots \end{aligned} \tag{3.2.8}$$

Similarly, by using the expanstions (3.2.4) and (3.2.3) into the constitutive equation (3.1.4), we obtain

$$\begin{aligned} & \frac{1}{\varepsilon^2}\chi^{-2}_{ijk}(x,y) + \frac{1}{\varepsilon}\chi^{-1}_{ijk}(x,y) + \chi^0_{ijk}(x,y) + \varepsilon\chi^1_{ijk}(x,y) + \cdots \\ = & \frac{1}{\varepsilon^2} b_{ijkhmn}(\frac{\partial}{\partial y_h}(e_{ymn}(u^0))) \\ & + \frac{1}{\varepsilon} b_{ijkhmn}(\frac{\partial}{\partial x_h}(e_{ymn}(u^0)) + \frac{\partial}{\partial y_h}(e_{xmn}(u^0)) + \frac{\partial}{\partial y_h}(e_{ymn}(u^1))) \\ & + b_{ijkhmn}(\frac{\partial}{\partial x_h}(e_{xmn}(u^0)) + \frac{\partial}{\partial x_h}(e_{ymn}(u^1)) + \frac{\partial}{\partial y_h}(e_{xmn}(u^1)) + \frac{\partial}{\partial y_h}(e_{ymn}(u^2))) \\ & + \varepsilon b_{ijkhmn}(\frac{\partial}{\partial x_h}(e_{xmn}(u^1)) + \frac{\partial}{\partial x_h}(e_{ymn}(u^2)) + \frac{\partial}{\partial y_h}(e_{xmn}(u^2)) + \frac{\partial}{\partial y_h}(e_{ymn}(u^3))) + \cdots \end{aligned} \tag{3.2.9}$$

Using the expanstion (3.2.3) into the equation of boundary condition (3.1.6) on Γ, we get

$$\frac{1}{\varepsilon^2}\chi_{ijk}^{-2}n_k + \frac{1}{\varepsilon}\chi_{ijk}^{-1}n_k + \chi_{ijk}^{0}n_k + \varepsilon\chi_{ijk}^{1}n_k + \varepsilon^2\chi_{ijk}^{2}n_k + \cdots = 0 \tag{3.2.10}$$

Using the expanstions (3.2.6) and (3.2.2) into the equation of boundary condition (3.1.7) on Γ, we have

$$-\frac{1}{\varepsilon^3}T_{ij}^{-3}N_j - \frac{1}{\varepsilon^2}T_{ij}^{-2}N_j + \frac{1}{\varepsilon}(\sigma_{ij}^{-1} - T_{ij}^{-1})N_j + (\sigma_{ij}^{0} - T_{ij}^{0})N_j + \varepsilon(\sigma_{ij}^{1} - T_{ij}^{1})N_j + \cdots = 0 \tag{3.2.11}$$

The periodic Ω is illustrated in Fig. 3.2.1, which is inside the sample. Γ and $\partial\Omega_S \cap \partial\Omega$ are boundaries of Ω_S . There are six normal unit vectors pointing outwards $\overrightarrow{N^1}\,(\overrightarrow{n^1})$ on $\partial\Omega^{1+}$, $-\overrightarrow{N^1}\,(-\overrightarrow{n^1})$ on $\partial\Omega^{1-}$, $\overrightarrow{N^2}\,(\overrightarrow{n^2})$ on $\partial\Omega^{2+}$, $-\overrightarrow{N^2}\,(-\overrightarrow{n^2})$ on $\partial\Omega^{2-}$, $\overrightarrow{N^3}\,(\overrightarrow{n^3})$ on $\partial\Omega^{3+}$ and $-\overrightarrow{N^3}\,(-\overrightarrow{n^3})$ on $\partial\Omega^{3-}$.

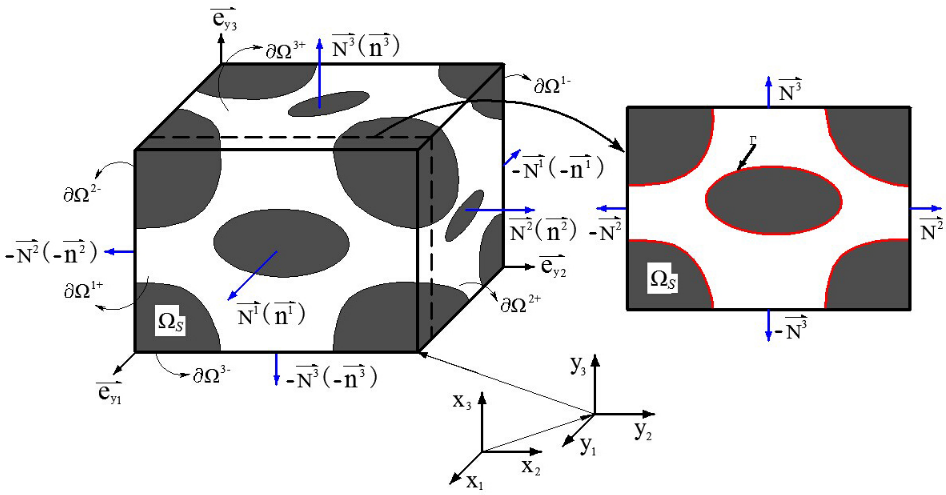

Figure 3.2.1: A periodic Ω of the empty porous matrix in the sample

We obtain a lowest order a boundary value problem for u^0:

in solid Ω_S,

$$\frac{\partial T_{ij}^{-3}}{\partial y_j} = 0 \tag{3.2.12}$$

$$T_{ij}^{-3} = \frac{\partial \chi_{ijk}^{-2}}{\partial y_k} \tag{3.2.13}$$

$$\chi_{ijk}^{-2} = b_{ijkhmn}(\frac{\partial}{\partial y_h}(e_{ymn}(u^0))) \tag{3.2.14}$$

boundary conditions on Γ,

$$\chi_{ijk}^{-2} n_k = 0 \qquad on \quad \Gamma \tag{3.2.15}$$

$$T_{ij}^{-3} N_j = 0 \qquad on \quad \Gamma \tag{3.2.16}$$

periodicity conditions on $\partial\Omega_S \cap \partial\Omega$,

$$\chi_{ijk}^{-2} @ n_k^1 = \Pi_{ij}^1, \quad T_{ij}^{-3} @ N_j^1 = F_i^1 \quad on \quad \partial\Omega_S \cap \partial\Omega^{1+} \tag{3.2.17}$$

$$\chi_{ijk}^{-2} @ (-n_k^1) = -\Pi_{ij}^1, \quad T_{ij}^{-3} @ (-N_j^1) = -F_i^1 \quad on \quad \partial\Omega_S \cap \partial\Omega^{1-} \tag{3.2.18}$$

$$\chi_{ijk}^{-2} @ n_k^2 = \Pi_{ij}^2, \quad T_{ij}^{-3} @ N_j^2 = F_i^2 \quad on \quad \partial\Omega_S \cap \partial\Omega^{2+} \tag{3.2.19}$$

$$\chi_{ijk}^{-2} @ (-n_k^2) = -\Pi_{ij}^2, \quad T_{ij}^{-3} @ (-N_j^2) = -F_i^2 \quad on \quad \partial\Omega_S \cap \partial\Omega^{2-} \tag{3.2.20}$$

$$\chi_{ijk}^{-2} @ n_k^3 = \Pi_{ij}^3, \quad T_{ij}^{-3} @ N_j^3 = F_i^3 \quad on \quad \partial\Omega_S \cap \partial\Omega^{3+} \tag{3.2.21}$$

$$\chi_{ijk}^{-2} @ (-n_k^3) = -\Pi_{ij}^3, \quad T_{ij}^{-3} @ (-N_j^3) = -F_i^3 \quad on \quad \partial\Omega_S \cap \partial\Omega^{3-} \tag{3.2.22}$$

We look for $\forall m, n,\ \frac{\partial u_m^0}{\partial y_n} = 0$ ($u^0 = u^0(x)$). The constitutive equations (3.2.13) and (3.2.14) become

$$\chi_{ijk}^{-2} = 0 \tag{3.2.23}$$

$$T_{ij}^{-3} = 0 \tag{3.2.24}$$

We can find balance equantion (3.2.12) and all boundary conditions which they are satisfied. They are possible to show the solutions $u^0 = u^0(x)$, $\chi_{ijk}^{-2} = 0$ and $T_{ij}^{-3} = 0$, in fact, unique.

To understand boundary condtion of T_{ij}^{-3} on Γ and on $\partial\Omega_S \cap \partial\Omega$, we need to introduce the weak formulation. Equation (3.2.12) is multiplied by any virtual velocity field v_i and then integrated over Ω with respect y. After integrating by parts and using the divergence theorem, we have

$$\forall v_i\,, \quad \int_{\Omega_S} \frac{\partial T_{ij}^{-3}}{\partial y_j} v_i dV = 0 \tag{3.2.25}$$

$$-\int_{\Omega_S} T_{ij}^{-3}\frac{\partial v_i}{\partial y_j}dV + \int_{\partial\Omega_S} T_{ij}^{-3}N_j v_i ds = 0 \quad (3.2.26)$$

where

$$\int_{\partial\Omega_S} T_{ij}^{-3}N_j v_i ds = \int_{\Gamma} T_{ij}^{-3}N_j v_i ds + \int_{\partial\Omega_S\cap\partial\Omega} T_{ij}^{-3}N_j v_i ds \quad (3.2.27)$$

we get $\int_\Gamma T_{ij}^{-3}N_j v_i ds = 0$ because $T_{ij}^{-3}N_j = 0$ on Γ.

$$\begin{aligned}\int_{\partial\Omega_S\cap\partial\Omega} T_{ij}^{-3}N_j v_i ds \;=\; & \int_{\partial\Omega_S\cap\partial\Omega^{1+}} (T_{ij}^{-3}@N_j^1)\bullet v_i ds + \int_{\partial\Omega_S\cap\partial\Omega^{1-}} (T_{ij}^{-3}@(-N_j^1))\bullet v_i ds \\ & + \int_{\partial\Omega_S\cap\partial\Omega^{2+}} (T_{ij}^{-3}@N_j^2)\bullet v_i ds + \int_{\partial\Omega_S\cap\partial\Omega^{2-}} (T_{ij}^{-3}@(-N_j^2))\bullet v_i ds \\ & + \int_{\partial\Omega_S\cap\partial\Omega^{3+}} (T_{ij}^{-3}@N_j^3)\bullet v_i ds + \int_{\partial\Omega_S\cap\partial\Omega^{3-}} (T_{ij}^{-3}@(-N_j^3))\bullet v_i ds \quad (3.2.28)\end{aligned}$$

Using the periodicity conditions on $\partial\Omega_S\cap\partial\Omega$, we get

$$\int_{\partial\Omega_S\cap\partial\Omega} T_{ij}^{-3}N_j v_i ds = F_i^1 - F_i^1 + F_i^2 - F_i^2 + F_i^3 - F_i^3 = 0 \quad (3.2.29)$$

$$\forall v_i\,,\quad \int_{\Omega_S} T_{ij}^{-3}\frac{\partial v_i}{\partial y_j}dV = 0 \quad (3.2.30)$$

Similarly, to understand boundary condtions of χ_{ijk}^{-2} on Γ and on $\partial\Omega_S\cap\partial\Omega$, we use the same way for equation (3.2.13)

$$\forall E_{ij}^*\,,\quad \int_{\Omega_S} \frac{\partial \chi_{ijk}^{-2}}{\partial y_k}E_{ij}^* dV - \int_{\Omega_S} T_{ij}^{-3}E_{ij}^* dV = 0 \quad (3.2.31)$$

$$-\int_{\Omega_S} \chi_{ijk}^{-2}\frac{\partial E_{ij}^*}{\partial y_k}dV + \int_{\partial\Omega_S} \chi_{ijk}^{-2}n_k E_{ij}^* ds - \int_{\Omega_S} T_{ij}^{-3}E_{ij}^* dV = 0 \quad (3.2.32)$$

$$\int_{\partial\Omega_S} \chi_{ijk}^{-2}n_k E_{ij}^* ds = \int_{\Gamma} \chi_{ijk}^{-2}n_k E_{ij}^* ds + \int_{\partial\Omega_S\cap\partial\Omega} \chi_{ijk}^{-2}n_k E_{ij}^* ds \quad (3.2.33)$$

Using $\chi_{ijk}^{-2}n_k = 0$ on Γ and periodicity conditions on $\partial\Omega_S\cap\partial\Omega$, we have that

$$\int_{\partial\Omega_S} \chi_{ijk}^{-2}n_k E_{ij}^* ds = 0 \quad (3.2.34)$$

$$\forall E_{ij}^{*}, \quad \int_{\Omega_S} \chi_{ijk}^{-2} \frac{\partial E_{ij}^{*}}{\partial y_k} dV + \int_{\Omega_S} T_{ij}^{-3} E_{ij}^{*} dV = 0 \tag{3.2.35}$$

At the following order, we obtain a boundary value problem for u^1 that

in solid Ω_S,

$$\frac{\partial T_{ij}^{-3}}{\partial x_j} + \frac{\partial T_{ij}^{-2}}{\partial y_j} = 0 \tag{3.2.36}$$

$$T_{ij}^{-3} = \frac{\partial \chi_{ijk}^{-2}}{\partial y_k} \tag{3.2.37}$$

$$T_{ij}^{-2} = \frac{\partial \chi_{ijk}^{-2}}{\partial x_k} + \frac{\partial \chi_{ijk}^{-1}}{\partial y_k} \tag{3.2.38}$$

$$\chi_{ijk}^{-1} = b_{ijkhmn}\left(\frac{\partial}{\partial x_h}(e_{ymn}(u^0)) + \frac{\partial}{\partial y_h}(e_{xmn}(u^0)) + \frac{\partial}{\partial y_h}(e_{ymn}(u^1))\right) \tag{3.2.39}$$

on boundary Γ,

$$\chi_{ijk}^{-1} n_k = 0 \qquad on \quad \Gamma \tag{3.2.40}$$

$$T_{ij}^{-2} N_j = 0 \qquad on \quad \Gamma \tag{3.2.41}$$

and the periodicity conditions on $\partial\Omega_S \cap \partial\Omega$

Using the results of $\chi_{ijk}^{-2} = 0$, $u^0 = u^0(x)$, and $T_{ij}^{-3} = 0$, we obtain that

in solid Ω_S,

$$\frac{\partial T_{ij}^{-2}}{\partial y_j} = 0 \tag{3.2.42}$$

$$T_{ij}^{-2} = \frac{\partial \chi_{ijk}^{-1}}{\partial y_k} \tag{3.2.43}$$

$$\chi_{ijk}^{-1} = b_{ijkhmn} \frac{\partial}{\partial y_h}(e_{ymn}(u^1)) \tag{3.2.44}$$

on boundaryΓ,

$$\chi_{ijk}^{-1} n_k = 0 \qquad on \quad \Gamma \tag{3.2.45}$$

$$T_{ij}^{-2} N_j = 0 \qquad on \quad \Gamma \tag{3.2.46}$$

and the periodicity conditions on $\partial\Omega_S \cap \partial\Omega$

We obtain a same form as the first order and we also get solutions $u^1 = u^1(x)$, $\chi_{ijk}^{-1} = 0$ and $T_{ij}^{-2} = 0$.

At the following order, we obtain a boundary value problem for u^2:

in solid Ω_S,

$$\frac{\partial T_{ij}^{-2}}{\partial x_j} - \frac{\partial \sigma_{ij}^{-1}}{\partial y_j} + \frac{\partial T_{ij}^{-1}}{\partial y_j} = 0 \tag{3.2.47}$$

$$T_{ij}^{-1} = \frac{\partial \chi_{ijk}^{-1}}{\partial x_k} + \frac{\partial \chi_{ijk}^{0}}{\partial y_k} \tag{3.2.48}$$

$$\chi_{ijk}^{0} = b_{ijkhmn}\left(\frac{\partial}{\partial x_h}(e_{xmn}(u^0)) + \frac{\partial}{\partial x_h}(e_{ymn}(u^1)) + \frac{\partial}{\partial y_h}(e_{xmn}(u^1)) + \frac{\partial}{\partial y_h}(e_{ymn}(u^2))\right) \tag{3.2.49}$$

$$\sigma_{ij}^{-1}(x,y) = a_{ijmn} e_{ymn}(u^0) \tag{3.2.50}$$

on boundary Γ,

$$\chi_{ijk}^{0} n_k = 0 \qquad on \quad \Gamma \tag{3.2.51}$$

$$(\sigma_{ij}^{-1} - T_{ij}^{-1}) N_j = 0 \qquad on \quad \Gamma \tag{3.2.52}$$

and the periodicity conditions on $\partial\Omega_S \cap \partial\Omega$

By using $\chi_{ijk}^{-1} = 0$, $\chi_{ijk}^{-2} = 0$, $u^0 = u^0(x)$, $u^1 = u^1(x)$, we obtain

in solid Ω_S,

$$\frac{\partial T_{ij}^{-1}}{\partial y_j} = 0 \tag{3.2.53}$$

$$T_{ij}^{-1} = \frac{\partial \chi_{ijk}^{0}}{\partial y_k} \tag{3.2.54}$$

$$\chi^0_{ijk} = b_{ijkhmn}(\frac{\partial}{\partial x_h}(e_{xmn}(u^0)) + \frac{\partial}{\partial y_h}(e_{ymn}(u^2))) \tag{3.2.55}$$

on boundary Γ,

$$\chi^0_{ijk} n_k = 0 \qquad on \quad \Gamma \tag{3.2.56}$$

$$T^{-1}_{ij} N_j = 0 \qquad on \quad \Gamma \tag{3.2.57}$$

and the periodicity conditions on $\partial\Omega_S \cap \partial\Omega$

The unknown u^2 is a linear vectorial function of $\frac{\partial}{\partial x}(e_x(u^0))$ and we enable to look for u^2 in form:

$$u^2_p = \xi^{qrs}_p \frac{\partial}{\partial x_q}(e_{xrs}(u^0)) + \overline{u}^2_p(x) \tag{3.2.58}$$

The χ^0_{ijk} and T^{-1}_{ij} can be written:

$$\begin{aligned}
\chi^0_{ijk} &= b_{ijkhmn}(\frac{\partial}{\partial x_h}(e_{xmn}(u^0)) + \frac{\partial}{\partial y_h}(e_{ymn}(u^2))) \\
&= b_{ijkhmn}(\frac{\partial}{\partial x_h}(e_{xmn}(u^0)) + \frac{\partial}{\partial y_h}(e_{ymn}(\xi^{qrs}))\frac{\partial}{\partial x_q}(e_{xrs}(u^0))) \\
&= (b_{ijkhmn} + b_{ijkqrs}\frac{\partial}{\partial y_q}(e_{yrs}(\xi^{hmn})))\frac{\partial}{\partial x_h}(e_{xmn}(u^0))
\end{aligned} \tag{3.2.59}$$

$$T^{-1}_{ij} = \frac{\partial \chi^0_{ijk}}{\partial y_k} = \frac{\partial(b_{ijkhmn} + b_{ijkqrs}\frac{\partial}{\partial y_q}(e_{yrs}(\xi^{hmn})))}{\partial y_k}\frac{\partial}{\partial x_h}(e_{xmn}(u^0)) \tag{3.2.60}$$

we shall check the equation $\frac{\partial T^{-1}_{ij}}{\partial y_j}$,

$$\frac{\partial T^{-1}_{ij}}{\partial y_j} = \frac{\partial^2(b_{ijkhmn} + b_{ijkqrs}\frac{\partial}{\partial y_q}(e_{yrs}(\xi^{hmn})))}{\partial y_j \partial y_k}\frac{\partial}{\partial x_h}(e_{xmn}(u^0)) \tag{3.2.61}$$

If the function ξ^{hmn} satisfies the equation,

$$\frac{\partial^2(b_{ijkhmn} + b_{ijkqrs}\frac{\partial}{\partial y_q}(e_{yrs}(\xi^{hmn})))}{\partial y_j \partial y_k} = 0 \tag{3.2.62}$$

we obtain

$$\frac{\partial T_{ij}^{-1}}{\partial y_j} = 0 \tag{3.2.63}$$

Using χ_{ijk}^0 to check the equation $\chi_{ijk}^0 n_k$, it is written

$$(b_{ijkhmn} + b_{ijkqrs}\frac{\partial}{\partial y_q}(e_{yrs}(\xi^{hmn})))\frac{\partial}{\partial x_h}(e_{xmn}(u^0))n_k \tag{3.2.64}$$

If the function ξ^{hmn} satisfies the equation,

$$(b_{ijkhmn} + b_{ijkqrs}\frac{\partial}{\partial y_q}(e_{yrs}(\xi^{hmn})))n_k = 0 \tag{3.2.65}$$

we get $\chi_{ijk}^0 n_k = 0$ on Γ.

we check to the equation $T_{ij}^{-1} N_j$ and we get

$$\frac{\partial(b_{ijkhmn} + b_{ijkqrs}\frac{\partial}{\partial y_q}(e_{yrs}(\xi^{hmn})))}{\partial y_k}\frac{\partial}{\partial x_h}(e_{xmn}(u^0))N_j \tag{3.2.66}$$

If the function ξ^{hmn} satisfies the equation,

$$\frac{\partial(b_{ijkhmn} + b_{ijkqrs}\frac{\partial}{\partial y_q}(e_{yrs}(\xi^{hmn})))}{\partial y_k} = 0 \tag{3.2.67}$$

we get $T_{ij}^{-1} N_j = 0$ on Γ.

Summary, ξ^{hmn} is solution of

$$\frac{\partial^2(b_{ijkhmn} + b_{ijkqrs}\frac{\partial}{\partial y_q}(e_{yrs}(\xi^{hmn})))}{\partial y_j \partial y_k} = 0 \tag{3.2.68}$$

$$\frac{\partial(b_{ijkhmn} + b_{ijkqrs}\frac{\partial}{\partial y_q}(e_{yrs}(\xi^{hmn})))}{\partial y_k} = 0 \qquad on \quad \Gamma \tag{3.2.69}$$

$$(b_{ijkhmn} + b_{ijkqrs}\frac{\partial}{\partial y_q}(e_{yrs}(\xi^{hmn})))n_k = 0 \qquad on \quad \Gamma \tag{3.2.70}$$

u^0, χ_{ijk}^0 and T_{ij}^{-1} are possible solutions, in fact, which are unique.

At the following order, we obtain a boundary value problem for u^3:

in solid Ω_S,

$$\frac{\partial \sigma_{ij}^{-1}}{\partial x_j} - \frac{\partial T_{ij}^{-1}}{\partial x_j} + \frac{\partial \sigma_{ij}^{0}}{\partial y_j} - \frac{\partial T_{ij}^{0}}{\partial y_j} = 0 \tag{3.2.71}$$

$$T_{ij}^{0} = \frac{\partial \chi_{ijk}^{0}}{\partial x_k} + \frac{\partial \chi_{ijk}^{1}}{\partial y_k} \tag{3.2.72}$$

$$\chi_{ijk}^{1} = b_{ijkhmn}\left(\frac{\partial}{\partial x_h}(e_{xmn}(u^1)) + \frac{\partial}{\partial x_h}(e_{ymn}(u^2)) + \frac{\partial}{\partial y_h}(e_{xmn}(u^2)) + \frac{\partial}{\partial y_h}(e_{ymn}(u^3))\right) \tag{3.2.73}$$

$$\sigma_{ij}^{0}(x,y) = a_{ijmn}(e_{xmn}(u^0) + e_{ymn}(u^1)) \tag{3.2.74}$$

on boundary Γ,

$$\chi_{ijk}^{1} n_k = 0 \qquad on \quad \Gamma \tag{3.2.75}$$

$$(\sigma_{ij}^{0} - T_{ij}^{0})N_j = 0 \qquad on \quad \Gamma \tag{3.2.76}$$

and the periodicity conditions on $\partial\Omega_S \cap \partial\Omega$

Using $u^1 = u^1(x)$, we obtain

$$\sigma_{ij}^{0}(x,y) = a_{ijmn} e_{xmn}(u^0) \tag{3.2.77}$$

Finally, a boundary value problem for u^4 is obtained at the following order

in solid Ω_S,

$$\frac{\partial \sigma_{ij}^{0}}{\partial x_j} - \frac{\partial T_{ij}^{0}}{\partial x_j} + \frac{\partial \sigma_{ij}^{1}}{\partial y_j} - \frac{\partial T_{ij}^{1}}{\partial y_j} = 0 \tag{3.2.78}$$

$$T_{ij}^{0} = \frac{\partial \chi_{ijk}^{0}}{\partial x_k} + \frac{\partial \chi_{ijk}^{1}}{\partial y_k} \tag{3.2.79}$$

on boundary Γ,

$$\chi_{ijk}^{1} n_k = 0 \qquad on \quad \Gamma \tag{3.2.80}$$

$$(\sigma_{ij}^{1} - T_{ij}^{1})N_j = 0 \qquad on \quad \Gamma \tag{3.2.81}$$

periodicity conditions on $\partial\Omega_S \cap \partial\Omega$,

$$\chi^1_{ijk}@n^1_k = \Pi^4_{ij}, \quad (\sigma^1_{ij} - T^1_{ij})@N^1_j = F^4_i, \quad on \quad \partial\Omega_S \cap \partial\Omega^{1+} \tag{3.2.82}$$

$$\chi^1_{ijk}@(-n^1_k) = -\Pi^4_{ij}, \quad (\sigma^1_{ij} - T^1_{ij})@(-N^1_j) = -F^4_i, \quad on \quad \partial\Omega_S \cap \partial\Omega^{1-} \tag{3.2.83}$$

$$\chi^1_{ijk}@n^2_k = \Pi^5_{ij}, \quad (\sigma^1_{ij} - T^1_{ij})@N^2_j = F^5_i, \quad on \quad \partial\Omega_S \cap \partial\Omega^{2+} \tag{3.2.84}$$

$$\chi^1_{ijk}@(-n^2_k) = -\Pi^5_{ij}, \quad (\sigma^1_{ij} - T^1_{ij})@(-N^2_j) = -F^5_i, \quad on \quad \partial\Omega_S \cap \partial\Omega^{2-} \tag{3.2.85}$$

$$\chi^1_{ijk}@n^3_k = \Pi^6_{ij}, \quad (\sigma^1_{ij} - T^1_{ij})@N^3_j = F^6_i, \quad on \quad \partial\Omega_S \cap \partial\Omega^{3+} \tag{3.2.86}$$

$$\chi^1_{ijk}@(-n^3_k) = -\Pi^6_{ij}, \quad (\sigma^1_{ij} - T^1_{ij})@(-N^3_j) = -F^6_i, \quad on \quad \partial\Omega_S \cap \partial\Omega^{3-} \tag{3.2.87}$$

Averaging equation (3.2.78) in volume Ω with respect to y and using the divergence theorem , we have

$$\begin{aligned} &< \frac{\partial\sigma^0_{ij}}{\partial x_j} > - < \frac{\partial T^0_{ij}}{\partial x_j} > = - \mid \Omega \mid^{-1} \int_{\Omega_S} (\frac{\partial\sigma^1_{ij}}{\partial y_j} - \frac{\partial T^1_{ij}}{\partial y_j}) dV = - \mid \Omega \mid^{-1} \int_{\partial\Omega_S} (\sigma^1_{ij} - T^1_{ij}) N_j ds \\ &= - \mid \Omega \mid^{-1} (\int_{\Gamma} (\sigma^1_{ij} - T^1_{ij}) N_j ds + \int_{\partial\Omega_S \cap \partial\Omega} (\sigma^1_{ij} - T^1_{ij}) N_j ds) \end{aligned} \tag{3.2.88}$$

Using $\int_{\partial\Omega_S \cap \partial\Omega} (\sigma^1_{ij} - T^1_{ij}) N_j ds = 0$ by periodicity conditions on $\partial\Omega_S \cap \partial\Omega$ and using the boundary conditions $(\sigma^1_{ij} - T^1_{ij}) N_j = 0$ on Γ , we get

$$< \frac{\partial\sigma^0_{ij}}{\partial x_j} > - < \frac{\partial T^0_{ij}}{\partial x_j} > = 0 \tag{3.2.89}$$

Exchanging order of integration and derivative, we get

$$< \frac{\partial\sigma^0_{ij}}{\partial x_j} > - < \frac{\partial T^0_{ij}}{\partial x_j} > = \frac{\partial < \sigma^0_{ij} >}{\partial x_j} - \frac{\partial < T^0_{ij} >}{\partial x_j} = 0 \tag{3.2.90}$$

We average equation (3.2.79) in volume Ω with respect to y and use the divergence theorem , we have

$$< T^0_{ij} >=< \frac{\partial \chi^0_{ijk}}{\partial x_k} > + \mid \Omega \mid^{-1} \int_{\Omega_S} \frac{\partial \chi^1_{ijk}}{\partial y_k} dV =< \frac{\partial \chi^0_{ijk}}{\partial x_k} > + \mid \Omega \mid^{-1} \int_{\partial \Omega_S} \chi^1_{ijk} n_k ds$$

$$=< \frac{\partial \chi^0_{ijk}}{\partial x_k} > + \mid \Omega \mid^{-1} \int_{\Gamma} \chi^1_{ijk} n_k ds + \mid \Omega \mid^{-1} \int_{\partial \Omega_S \cap \partial \Omega} \chi^1_{ijk} n_k ds \tag{3.2.91}$$

We have$\mid \Omega \mid^{-1} \int_{\partial \Omega_S} \chi^1_{ijk} n_k ds = 0$ by periodicity conditions on $\partial \Omega_S \cap \partial \Omega$ and $\chi^1_{ijk} n_k = 0$ on Γ and . We get

$$< T^0_{ij} >=< \frac{\partial \chi^0_{ijk}}{\partial x_k} >= \frac{\partial < \chi^0_{ijk} >}{\partial x_k} \tag{3.2.92}$$

We obtain macroscpic second grade modelling of empty porous matrix by homogenization

$$\frac{\partial < \sigma^0_{ij} >}{\partial x_j} - \frac{\partial < T^0_{ij} >}{\partial x_j} = 0 \tag{3.2.93}$$

$$< T^0_{ij} >= \frac{\partial < \chi^0_{ijk} >}{\partial x_k} \tag{3.2.94}$$

$$< \sigma^0_{ij} >=< a_{ijmn} > e_{xmn}(u^0) \tag{3.2.95}$$

$$< \chi^0_{ijk} >=< b_{ijkhmn} + b_{ijkqrs} \frac{\partial}{\partial y_q}(e_{yrs}(\xi^{hmn})) > \frac{\partial}{\partial x_h}(e_{xmn}(u^0)) \tag{3.2.96}$$

Chapter 4

Second grade modelling of saturated porous matrix

4.1 Saturated porous matrix description

We are investigating the macroscopic behaviour of a saturated porous matrix which contains a filtrating fluid when the movement is quasi-static: velocities are small and accelerations are negligible. Both solid and fliud parts are connected. For the sake of simplicity, we adopt the following hypotheses: (1) the material comprising of the porous matrix is same that was described in chapter 3; (2) the fluid is viscous Newtonian and incompressible; (3) the viscosity is a constant. The saturated porous matrix is illustrated (Fig. 4.1.1) which consists of solid Φ_S, fluid Φ_F and interface Γ.

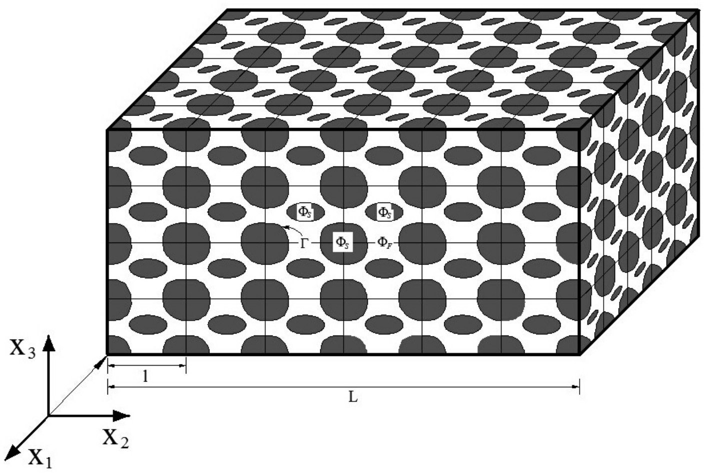

Figure 4.1.1: Periodic saturated porous matrix

The pore scale behaviour is same the empty porous matrix in solid Φ_S, and is Navier-Stokes equation the fluid Φ_F. The viscosity coefficient is $\mu\varepsilon^2$ because it represents a biphasic macroscopic behaviour. In other words, the fluid can only flow when the viscosity coefficient is small. The pore scale behaviour is as follows:

in solid Φ_S,

$$\frac{\partial}{\partial x_j}(\sigma^{\varepsilon}_{Sij} - T^{\varepsilon}_{Sij}) = 0 \tag{4.1.1}$$

$$\sigma^{\varepsilon}_{Sij}(x,t) = a_{ijmn} e_{xmn}(u^{\varepsilon}_S) \tag{4.1.2}$$

$$T^{\varepsilon}_{Sij}(x,t) = \frac{\partial \chi^{\varepsilon}_{Sijk}}{\partial x_k} \tag{4.1.3}$$

$$\chi^{\varepsilon}_{Sijk}(x,t) = b_{ijkhmn} \frac{\partial e_{xmn}(u^{\varepsilon}_S)}{\partial x_h} \tag{4.1.4}$$

$$e_{xmn}(u^{\varepsilon}_S) = \frac{1}{2}(\frac{\partial u^{\varepsilon}_m}{\partial x_n} + \frac{\partial u^{\varepsilon}_n}{\partial x_m}) \tag{4.1.5}$$

in fluid Φ_F,

$$\frac{\partial \sigma^{\varepsilon}_{Fij}}{\partial x_j} = 0 \tag{4.1.6}$$

$$\sigma^{\varepsilon}_{Fij}(x,t) = 2\mu\varepsilon^2 D^{\varepsilon}_{ij} - p^{\varepsilon} I_{ij} \tag{4.1.7}$$

$$D^{\varepsilon}_{ij} = \frac{1}{2}(\frac{\partial v^{\varepsilon}_{Fi}}{\partial x_j} + \frac{\partial v^{\varepsilon}_{Fj}}{\partial x_i}) \tag{4.1.8}$$

and the incompressibility condition

$$\frac{\partial v^{\varepsilon}_{Fi}}{\partial x_i} = 0 \tag{4.1.9}$$

on interface Γ,

$$\chi^{\varepsilon}_{Sijk} n_k = 0 \qquad on \quad \Gamma \tag{4.1.10}$$

$$(\sigma^{\varepsilon}_{Sij} - T^{\varepsilon}_{Sij}) N_j = \sigma^{\varepsilon}_{Fij} N_j \qquad on \quad \Gamma \tag{4.1.11}$$

$$v_{Si}^{\varepsilon} = v_{Fi}^{\varepsilon} \qquad on \quad \Gamma \tag{4.1.12}$$

and boundary conditions outside the sample (unspecified here)

4.2 Second grade modelling of saturated porous matrix

We look for u_S^{ε}, v_F^{ε} and p^{ε} by the double scale expansion in the form:

$$u_S^{\varepsilon}(x,t) = u_S^0(x,\frac{x}{\varepsilon},t) + \varepsilon u_S^1(x,\frac{x}{\varepsilon},t) + \varepsilon^2 u_S^2(x,\frac{x}{\varepsilon},t) + \cdots \tag{4.2.1}$$

$$v_F^{\varepsilon}(x,t) = v_F^0(x,\frac{x}{\varepsilon},t) + \varepsilon v_F^1(x,\frac{x}{\varepsilon},t) + \varepsilon^2 v_F^2(x,\frac{x}{\varepsilon},t) + \cdots \tag{4.2.2}$$

$$p^{\varepsilon}(x,t) = p^0(x,\frac{x}{\varepsilon},t) + \varepsilon p^1(x,\frac{x}{\varepsilon},t) + \varepsilon^2 p^2(x,\frac{x}{\varepsilon},t) + \cdots \tag{4.2.3}$$

and the form of stress $\sigma_{Sij}^{\varepsilon}(x)$, $\sigma_{Fij}^{\varepsilon}(x)$ and second grade stress $\chi_{Sijk}^{\varepsilon}(x)$ for the same reason as the 1D periodic medium,

$$\sigma_{Fij}^{\varepsilon}(x,t) = \sigma_{Fij}^0(x,\frac{x}{\varepsilon},t) + \varepsilon \sigma_{Fij}^1(x,\frac{x}{\varepsilon},t) + \varepsilon^2 \sigma_{Fij}^2(x,\frac{x}{\varepsilon},t) + \cdots \tag{4.2.4}$$

$$\sigma_{Sij}^{\varepsilon}(x,t) = \frac{1}{\varepsilon}\sigma_{Sij}^{-1}(x,\frac{x}{\varepsilon},t) + \sigma_{Sij}^0(x,\frac{x}{\varepsilon},t) + \varepsilon \sigma_{Sij}^1(x,\frac{x}{\varepsilon},t) + \varepsilon^2 \sigma_{Sij}^2(x,\frac{x}{\varepsilon},t) + \cdots \tag{4.2.5}$$

$$\chi_{Sijk}^{\varepsilon}(x,t) = \frac{1}{\varepsilon^2}\chi_{Sijk}^{-2}(x,\frac{x}{\varepsilon},t) + \frac{1}{\varepsilon}\chi_{Sijk}^{-1}(x,\frac{x}{\varepsilon},t) + \chi_{Sijk}^0(x,\frac{x}{\varepsilon},t) + \varepsilon \chi_{Sijk}^1(x,\frac{x}{\varepsilon},t) + \cdots \tag{4.2.6}$$

with $y = x/\varepsilon$ and where u_S^i, v_F^i, σ_{Sij}^i, σ_{Fij}^i, χ_{Sijk}^i and p^i are y-periodic, with period Ω. Not that we have introduced a possible time dependence which could be due to fliud volume change.

We analysis the pore scale behaviour by the double scale asymptotic expansions technique. It is same as the empty porous matrix in solid Φ_S.

$$e_{xmn}(u_S^{\varepsilon}) = \frac{1}{\varepsilon}e_{ymn}(u_S^0) + e_{xmn}(u_S^0) + e_{ymn}(u_S^1) + \varepsilon(e_{xmn}(u_S^1) + e_{ymn}(u_S^2)) + \cdots \tag{4.2.7}$$

$$T_{Sij}^{\varepsilon}(x,t) = \frac{\partial \chi_{Sijk}^{\varepsilon}}{\partial x_k} = \frac{1}{\varepsilon^3}\frac{\partial \chi_{Sijk}^{-2}}{\partial y_k} + \frac{1}{\varepsilon^2}(\frac{\partial \chi_{Sijk}^{-2}}{\partial x_k} + \frac{\partial \chi_{Sijk}^{-1}}{\partial y_k}) + \frac{1}{\varepsilon}(\frac{\partial \chi_{Sijk}^{-1}}{\partial x_k} + \frac{\partial \chi_{Sijk}^{0}}{\partial y_k}) + \cdots \tag{4.2.8}$$

We define

$$T^{\varepsilon}_{Sij}(x,t)=\frac{1}{\varepsilon^3}T^{-3}_{Sij}+\frac{1}{\varepsilon^2}T^{-2}_{Sij}+\frac{1}{\varepsilon}T^{-1}_{Sij}+T^{0}_{Sij}+\varepsilon T^{1}_{Sij}+\varepsilon^2T^{2}_{Sij}+\cdots \tag{4.2.9}$$

where $T^{-3}_{Sij}=\frac{\partial\chi^{-2}_{Sijk}}{\partial y_k}$, $T^{-2}_{Sij}=\frac{\partial\chi^{-2}_{Sijk}}{\partial x_k}+\frac{\partial\chi^{-1}_{Sijk}}{\partial y_k}$, $T^{-1}_{Sij}=\frac{\partial\chi^{-1}_{Sijk}}{\partial x_k}+\frac{\partial\chi^{0}_{Sijk}}{\partial y_k}$, $T^{0}_{Sij}=\frac{\partial\chi^{0}_{Sijk}}{\partial x_k}+\frac{\partial\chi^{1}_{Sijk}}{\partial y_k}\cdots$.
Using the expansions into the banlance equation (4.1.1), it is written

$$\begin{aligned}&-\frac{1}{\varepsilon^4}\frac{\partial T^{-3}_{Sij}}{\partial y_j}-\frac{1}{\varepsilon^3}(\frac{\partial T^{-3}_{Sij}}{\partial x_j}+\frac{\partial T^{-2}_{Sij}}{\partial y_j})-\frac{1}{\varepsilon^2}(\frac{\partial T^{-2}_{Sij}}{\partial x_j}-\frac{\partial \sigma^{-1}_{Sij}}{\partial y_j}+\frac{\partial T^{-1}_{Sij}}{\partial y_j})\\&+\frac{1}{\varepsilon}(\frac{\partial \sigma^{-1}_{Sij}}{\partial x_j}-\frac{\partial T^{-1}_{Sij}}{\partial x_j}+\frac{\partial \sigma^{0}_{Sij}}{\partial y_j}-\frac{\partial T^{0}_{Sij}}{\partial y_j})+(\frac{\partial \sigma^{0}_{Sij}}{\partial x_j}-\frac{\partial T^{0}_{Sij}}{\partial x_j}+\frac{\partial \sigma^{1}_{Sij}}{\partial y_j}-\frac{\partial T^{1}_{Sij}}{\partial y_j})+\cdots=0\end{aligned} \tag{4.2.10}$$

Using the expansions into the constitutive equation (4.1.2) , we obtain

$$\begin{aligned}&\frac{1}{\varepsilon}\sigma^{-1}_{Sij}(x,y,t)+\sigma^{0}_{Sij}(x,y,t)+\varepsilon\sigma^{1}_{Sij}(x,y,t)+\cdots\\=\;&\frac{1}{\varepsilon}a_{ijmn}e_{ymn}(u^0_S)+a_{ijmn}(e_{xmn}(u^0_S)+e_{ymn}(u^1_S))+\varepsilon a_{ijmn}(e_{xmn}(u^1_S)+e_{ymn}(u^2_S))+\cdots\end{aligned} \tag{4.2.11}$$

Using the expansions into the constitutive equation (4.1.4), the constitutive law becomes

$$\begin{aligned}&\frac{1}{\varepsilon^2}\chi^{-2}_{Sijk}(x,y,t)+\frac{1}{\varepsilon}\chi^{-1}_{Sijk}(x,y,t)+\chi^{0}_{Sijk}(x,y,t)+\varepsilon\chi^{1}_{Sijk}(x,y,t)+\cdots\\=\;&\frac{1}{\varepsilon^2}b_{ijkhmn}(\frac{\partial}{\partial y_h}(e_{ymn}(u^0_S)))\\&+\frac{1}{\varepsilon}b_{ijkhmn}(\frac{\partial}{\partial x_h}(e_{ymn}(u^0_S))+\frac{\partial}{\partial y_h}(e_{xmn}(u^0_S))+\frac{\partial}{\partial y_h}(e_{ymn}(u^1_S)))\\&+b_{ijkhmn}(\frac{\partial}{\partial x_h}(e_{xmn}(u^0_S))+\frac{\partial}{\partial x_h}(e_{ymn}(u^1_S))+\frac{\partial}{\partial y_h}(e_{xmn}(u^1_S))+\frac{\partial}{\partial y_h}(e_{ymn}(u^2_S)))\\&+\varepsilon b_{ijkhmn}(\frac{\partial}{\partial x_h}(e_{xmn}(u^1_S))+\frac{\partial}{\partial x_h}(e_{ymn}(u^2_S))+\frac{\partial}{\partial y_h}(e_{xmn}(u^2_S))+\frac{\partial}{\partial y_h}(e_{ymn}(u^3_S)))+\cdots\end{aligned} \tag{4.2.12}$$

In fluid Φ_F, using the expansions (4.2.2) into the equations $\frac{\partial v^{\varepsilon}_{Fi}}{\partial x_j}$ and $\frac{\partial v^{\varepsilon}_{Fj}}{\partial x_i}$, we have

$$\frac{\partial v^{\varepsilon}_{Fi}}{\partial x_j}=\frac{1}{\varepsilon}\frac{\partial v^{0}_{Fi}}{\partial y_j}+\frac{\partial v^{0}_{Fi}}{\partial x_j}+\frac{\partial v^{1}_{Fi}}{\partial y_j}+\varepsilon(\frac{\partial v^{1}_{Fi}}{\partial x_j}+\frac{\partial v^{2}_{Fi}}{\partial y_j})+\cdots \tag{4.2.13}$$

$$\frac{\partial v^{\varepsilon}_{Fj}}{\partial x_i}=\frac{1}{\varepsilon}\frac{\partial v^{0}_{Fj}}{\partial y_i}+\frac{\partial v^{0}_{Fj}}{\partial x_i}+\frac{\partial v^{1}_{Fj}}{\partial y_i}+\varepsilon(\frac{\partial v^{1}_{Fj}}{\partial x_i}+\frac{\partial v^{2}_{Fj}}{\partial y_i})+\cdots \tag{4.2.14}$$

Hence, equation (4.1.8) is written

$$D_{ij}^{\varepsilon}=\frac{1}{\varepsilon}\frac{1}{2}(\frac{\partial v_{Fi}^{0}}{\partial y_{j}}+\frac{\partial v_{Fj}^{0}}{\partial y_{i}})+\frac{1}{2}(\frac{\partial v_{Fi}^{0}}{\partial x_{j}}+\frac{\partial v_{Fj}^{0}}{\partial x_{i}}+\frac{\partial v_{Fi}^{1}}{\partial y_{j}}+\frac{\partial v_{Fj}^{1}}{\partial y_{i}})+\cdots \tag{4.2.15}$$

Using the expansions (4.2.15) and (4.2.3) into equation (4.1.7), we obtain

$$\begin{aligned}&\sigma_{Fij}^{\varepsilon}=2\mu\varepsilon^{2}D_{ij}^{\varepsilon}-p^{\varepsilon}I_{ij}=\\&-p^{0}I_{ij}+\varepsilon(\mu(\frac{\partial v_{Fi}^{0}}{\partial y_{j}}+\frac{\partial v_{Fj}^{0}}{\partial y_{i}})-p^{1}I_{ij})+\varepsilon^{2}(\mu(\frac{\partial v_{Fi}^{0}}{\partial x_{j}}+\frac{\partial v_{Fj}^{0}}{\partial x_{i}}+\frac{\partial v_{Fi}^{1}}{\partial y_{j}}+\frac{\partial v_{Fj}^{1}}{\partial y_{i}})-p^{2}I_{ij})+\cdots\end{aligned} \tag{4.2.16}$$

Using the expansion (4.2.4) into eaquation (4.2.16), it is as a constitutive law for the fluid and it is written

$$\begin{aligned}&\sigma_{Fij}^{0}+\varepsilon\sigma_{Fij}^{1}+\varepsilon^{2}\sigma_{Fij}^{2}+\cdots\\=\;&-p^{0}I_{ij}+\varepsilon(\mu(\frac{\partial v_{Fi}^{0}}{\partial y_{j}}+\frac{\partial v_{Fj}^{0}}{\partial y_{i}})-p^{1}I_{ij})+\varepsilon^{2}(\mu(\frac{\partial v_{Fi}^{0}}{\partial x_{j}}+\frac{\partial v_{Fj}^{0}}{\partial x_{i}}+\frac{\partial v_{Fi}^{1}}{\partial y_{j}}+\frac{\partial v_{Fj}^{1}}{\partial y_{i}})-p^{2}I_{ij})+\cdots\end{aligned} \tag{4.2.17}$$

Using the expansion (4.2.4) into the balance equation (4.1.6) , we have

$$\frac{1}{\varepsilon}\frac{\partial\sigma_{Fij}^{0}}{\partial y_{j}}+(\frac{\partial\sigma_{Fij}^{0}}{\partial x_{j}}+\frac{\partial\sigma_{Fij}^{1}}{\partial y_{j}})+\varepsilon(\frac{\partial\sigma_{Fij}^{1}}{\partial x_{j}}+\frac{\partial\sigma_{Fij}^{2}}{\partial y_{j}})+\cdots=0 \tag{4.2.18}$$

Using the expansion (4.2.2) into the incompressibility condition equation (4.1.9) , we get

$$\frac{1}{\varepsilon}\frac{\partial v_{Fi}^{0}}{\partial y_{i}}+\frac{\partial v_{Fi}^{0}}{\partial x_{i}}+\frac{\partial v_{Fi}^{1}}{\partial y_{i}}+\varepsilon(\frac{\partial v_{Fi}^{1}}{\partial x_{i}}+\frac{\partial v_{Fi}^{2}}{\partial y_{i}})+\cdots=0 \tag{4.2.19}$$

On the interface Γ, using the expansion (4.2.6) into equation (4.1.10), we have

$$\frac{1}{\varepsilon^{2}}\chi_{Sijk}^{-2}n_{k}+\frac{1}{\varepsilon}\chi_{Sijk}^{-1}n_{k}+\chi_{Sijk}^{0}n_{k}+\varepsilon\chi_{Sijk}^{1}n_{k}+\cdots=0 \tag{4.2.20}$$

On the interface Γ, using the expantions (4.2.9) and (4.2.5) into equation (4.1.11), we get

$$\begin{aligned}&-\frac{1}{\varepsilon^{3}}T_{Sij}^{-3}N_{j}-\frac{1}{\varepsilon^{2}}T_{Sij}^{-2}N_{j}+\frac{1}{\varepsilon}(\sigma_{Sij}^{-1}-T_{Sij}^{-1})N_{j}+(\sigma_{Sij}^{0}-T_{Sij}^{0})N_{j}+\varepsilon(\sigma_{Sij}^{1}-T_{Sij}^{1})N_{j}+\cdots\\=\;&\sigma_{Fij}^{0}N_{j}+\varepsilon\sigma_{Fij}^{1}N_{j}+\cdots\end{aligned} \tag{4.2.21}$$

On the interface Γ, using the expantions (4.2.1) and (4.2.2) into equation (4.1.12), we obtain

$$\frac{\partial u_{si}^0}{\partial t} + \varepsilon \frac{\partial u_{si}^1}{\partial t} + \varepsilon^2 \frac{\partial u_{si}^2}{\partial t} + \cdots = v_{Fi}^0 + \varepsilon v_{Fi}^1 + \varepsilon^2 v_{Fi}^2 + \cdots \tag{4.2.22}$$

The periodic Ω is presented in Fig. 4.2.1 inside the sample. Γ and $\partial\Omega_S \cap \partial\Omega$ are boundaries of Ω_S and Γ and $\partial\Omega_F \cap \partial\Omega$ are boundaries of Ω_F. There are six normal unit vectors pointing outwards $\overrightarrow{N^1}\,(\overrightarrow{n^1})$ on $\partial\Omega^{1+}$, $-\overrightarrow{N^1}\,(-\overrightarrow{n^1})$ on $\partial\Omega^{1-}$, $\overrightarrow{N^2}\,(\overrightarrow{n^2})$ on $\partial\Omega^{2+}$, $-\overrightarrow{N^2}\,(-\overrightarrow{n^2})$ on $\partial\Omega^{2-}$, $\overrightarrow{N^3}\,(\overrightarrow{n^3})$ on $\partial\Omega^{3+}$ and $-\overrightarrow{N^3}\,(-\overrightarrow{n^3})$ on $\partial\Omega^{3-}$.

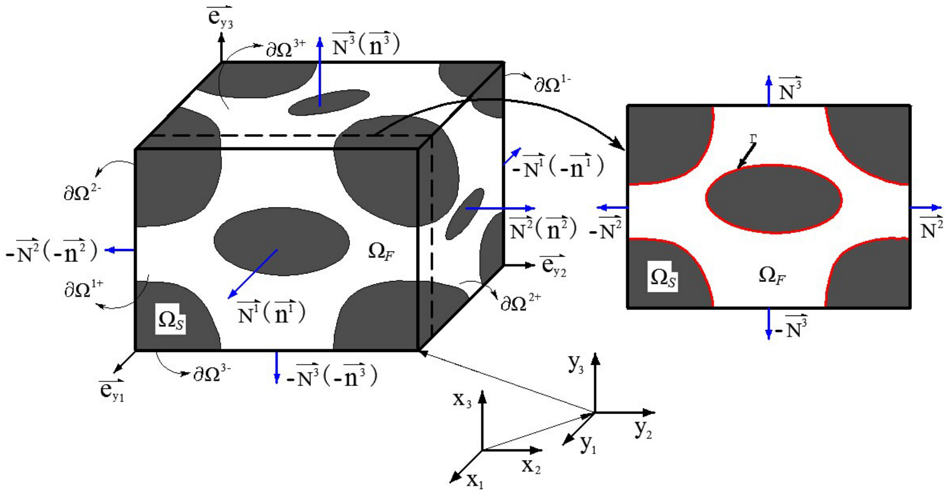

Figure 4.2.1: A periodic Ω of the saturated porous matrix in the sample

We obtain at the lowest order a interface value problem for u_S^0:

in solid Ω_S,

$$\frac{\partial T_{Sij}^{-3}}{\partial y_j} = 0 \tag{4.2.23}$$

$$T_{Sij}^{-3} = \frac{\partial \chi_{Sijk}^{-2}}{\partial y_k} \tag{4.2.24}$$

$$\chi_{Sijk}^{-2} = b_{ijkhmn}(\frac{\partial}{\partial y_h}(e_{ymn}(u_S^0))) \tag{4.2.25}$$

interface condition on Γ,

$$\chi_{Sijk}^{-2} n_k = 0 \qquad on \quad \Gamma \tag{4.2.26}$$

$$T_{Sij}^{-3} N_j = 0 \qquad on \quad \Gamma \tag{4.2.27}$$

and the periodicity conditions on $\partial\Omega_S \cap \partial\Omega$

The form is the same as in the empty porous medium. We get the solutions $u_S^0 = u_S^0(x,t)$, $\chi_{Sijk}^{-2} = 0$ and $T_{Sij}^{-3} = 0$.

At the following order, we solve similar equations for u_S^1 which are now written:

in solid Ω_S,

$$\frac{\partial T_{Sij}^{-2}}{\partial y_j} = 0 \tag{4.2.28}$$

$$T_{Sij}^{-2} = \frac{\partial \chi_{Sijk}^{-1}}{\partial y_k} \tag{4.2.29}$$

$$\chi_{Sijk}^{-1} = b_{ijkhmn} \frac{\partial}{\partial y_h}(e_{ymn}(u_S^1)) \tag{4.2.30}$$

on interface Γ,

$$\chi_{Sijk}^{-1} n_k = 0 \qquad on \quad \Gamma \tag{4.2.31}$$

$$T_{Sij}^{-2} N_j = 0 \qquad on \quad \Gamma \tag{4.2.32}$$

and the periodicity conditions on $\partial\Omega_S \cap \partial\Omega$

The form is the same as in the empty porous medium. We obtain solutions $u_S^1 = u_S^1(x,t)$, $\chi_{Sijk}^{-1} = 0$ and $T_{Sij}^{-2} = 0$.

At the following order, we obtain the interface value problem for u_S^2:

in solid Ω_S,

$$\frac{\partial T_{Sij}^{-1}}{\partial y_j} = 0 \tag{4.2.33}$$

$$T_{Sij}^{-1} = \frac{\partial \chi_{Sijk}^{0}}{\partial y_k} \tag{4.2.34}$$

$$\chi_{Sijk}^{0} = b_{ijkhmn}\left(\frac{\partial}{\partial x_h}(e_{xmn}(u_S^0)) + \frac{\partial}{\partial y_h}(e_{ymn}(u_S^2))\right) \tag{4.2.35}$$

on interface Γ,

$$\chi^0_{Sijk} n_k = 0 \qquad on \quad \Gamma \tag{4.2.36}$$

$$T^{-1}_{Sij} N_j = 0 \qquad on \quad \Gamma \tag{4.2.37}$$

and the periodicity conditions on $\partial\Omega_S \cap \partial\Omega$

The form is the same as in the empty porous medium and we also get the same solutions u^0_S, χ^0_{Sijk} and T^{-1}_{Sij} as in the empty porous mudia.

At the following order, we obtain the interface value problem for u^3_F and p^0:

in solid Ω_S,

$$\frac{\partial \sigma^{-1}_{Sij}}{\partial x_j} - \frac{\partial T^{-1}_{Sij}}{\partial x_j} + \frac{\partial \sigma^0_{Sij}}{\partial y_j} - \frac{\partial T^0_{Sij}}{\partial y_j} = 0 \tag{4.2.38}$$

$$\sigma^0_{Sij}(x,y) = a_{ijmn}(e_{xmn}(u^0_S) + e_{ymn}(u^1_S)) \tag{4.2.39}$$

in fluid Ω_F,

$$\frac{\partial \sigma^0_{Fij}}{\partial y_j} = 0 \tag{4.2.40}$$

$$\sigma^0_{Fij} = -p^0 I_{ij} \tag{4.2.41}$$

on interface Γ,

$$\chi^0_{Sijk} n_k = 0 \qquad on \quad \Gamma \tag{4.2.42}$$

$$(\sigma^0_{Sij} - T^0_{Sij}) N_j = \sigma^0_{Fij} N_j \qquad on \quad \Gamma \tag{4.2.43}$$

and the periodicity conditions on $\partial\Omega_S \cap \partial\Omega$

and the periodicity conditions on $\partial\Omega_F \cap \partial\Omega$

Using $u^1_S = u^1_S(x,t)$, we have

$$\sigma^0_{Sij}(x,y) = a_{ijmn} e_{xmn}(u^0_S) \tag{4.2.44}$$

and we obtain

$$p^0 = p^0(x,t) \tag{4.2.45}$$

In fluid Ω_F, we have the following order yields a interface value problem for v^0. Here, we prove the Darcy law and this work had done by Auriault (1983)[1].

$$\frac{\partial \sigma^0_{Fij}}{\partial x_j} + \frac{\partial \sigma^1_{Fij}}{\partial y_j} = 0 \tag{4.2.46}$$

$$\sigma^1_{Fij} = \mu(\frac{\partial v^0_{Fi}}{\partial y_j} + \frac{\partial v^0_{Fj}}{\partial y_i}) - p^1 I_{ij} \tag{4.2.47}$$

$$\frac{\partial v^0_{Fi}}{\partial y_i} = 0 \tag{4.2.48}$$

$$v^0_{Fi} = \frac{\partial u^0_{Si}}{\partial t} \qquad on \quad \Gamma \tag{4.2.49}$$

By using the relative velocity $\omega^0_i = v^0_{Fi} - \frac{\partial u^0_{Si}}{\partial t}$ and defining $\varepsilon^0_{yij} = \frac{1}{2}(\frac{\partial \omega^0_i}{\partial y_j} + \frac{\partial \omega^0_j}{\partial y_i})$, we have

$$\frac{\partial \sigma^0_{Fij}}{\partial x_j} + \frac{\partial \sigma^1_{Fij}}{\partial y_j} = 0 \tag{4.2.50}$$

$$\sigma^1_{Fij} = 2\mu\varepsilon^0_{yij} - p^1 I_{ij} \tag{4.2.51}$$

$$\frac{\partial \omega^0_i}{\partial y_i} = 0 \tag{4.2.52}$$

$$\omega^0_i = 0 \qquad on \quad \Gamma \tag{4.2.53}$$

The problem is unknown ω^0_i and p^1 and known p^1 with linear equations (4.2.50) and (4.2.53). Thus, we enable to look for p^1 the following form

$$p^1 = \beta^j(y)\frac{\partial p^0}{\partial x_j} + \overline{p}^1(x,t) \tag{4.2.54}$$

and ω^0_i is a linear vectorial function $\frac{\partial p^0}{\partial x_j}$

$$\omega_i^0 = k_{ij}\frac{\partial p^0}{\partial x_j} \tag{4.2.55}$$

and function k_{ij} is solution of:

$$\frac{\partial k_{ij}}{\partial y_i} = 0 \tag{4.2.56}$$

And the function k_{ij} satisfies the equation

$$k_{ij} = 0 \qquad on \quad \Gamma \tag{4.2.57}$$

ω_i^0, p^1 are the solutions, in fact, unique.

The mean of equation (4.2.55) yields the Darcy law:

$$< \omega_i^0 > = \mid \Omega \mid^{-1} \int_{\Omega_F} v_{Fi}^0 dV - \mid \Omega \mid^{-1} \int_{\Omega_S} \frac{\partial u_{Si}^0}{\partial t} dV = < v_{Fi}^0 > -\phi \frac{\partial u_{Si}^0}{\partial t} = \frac{1}{\mid \Omega \mid} \int_{\Omega_F} k_{ij} dV \frac{\partial p^0}{\partial x_j} \tag{4.2.58}$$

Finally, we obtain the intertace value problem for u_F^4:

in solid Ω_S,

$$\frac{\partial \sigma_{Sij}^0}{\partial x_j} - \frac{\partial T_{Sij}^0}{\partial x_j} + \frac{\partial \sigma_{Sij}^1}{\partial y_j} - \frac{\partial T_{Sij}^1}{\partial y_j} = 0 \tag{4.2.59}$$

$$T_{Sij}^0 = \frac{\partial \chi_{Sijk}^0}{\partial x_k} + \frac{\partial \chi_{Sijk}^1}{\partial y_k} \tag{4.2.60}$$

in fluid Ω_F,

$$\frac{\partial \sigma_{Fij}^0}{\partial x_j} + \frac{\partial \sigma_{Fij}^1}{\partial y_j} = 0 \tag{4.2.61}$$

$$\sigma_{Fij}^0 = -p^0 I_{ij} \tag{4.2.62}$$

on interface Γ,

$$\chi_{Sijk}^1 n_k = 0 \qquad on \quad \Gamma \tag{4.2.63}$$

$$(\sigma_{Sij}^1 - T_{Sij}^1) N_j = \sigma_{Fij}^1 N_j \qquad on \quad \Gamma \tag{4.2.64}$$

periodicity conditions on $\partial\Omega_S \cap \partial\Omega$,

$$\chi^1_{Sijk}@n^1_k = \Pi^1_{Sij}, \quad (\sigma^1_{Sij} - T^1_{Sij})@N^1_j = F^1_{Si}, \quad on \quad \partial\Omega_S \cap \partial\Omega^{1+} \tag{4.2.65}$$

$$\chi^1_{Sijk}@(-n^1_k) = -\Pi^1_{Sij}, \quad (\sigma^1_{Sij} - T^1_{Sij})@(-N^1_j) = -F^1_{Si}, \quad on \quad \partial\Omega_S \cap \partial\Omega^{1-} \tag{4.2.66}$$

$$\chi^1_{Sijk}@n^2_k = \Pi^2_{Sij}, \quad (\sigma^1_{Sij} - T^1_{Sij})@N^2_j = F^2_{Si}, \quad on \quad \partial\Omega_S \cap \partial\Omega^{2+} \tag{4.2.67}$$

$$\chi^1_{Sijk}@(-n^2_k) = -\Pi^2_{Sij}, \quad (\sigma^1_{Sij} - T^1_{Sij})@(-N^2_j) = -F^2_{Si}, \quad on \quad \partial\Omega_S \cap \partial\Omega^{2-} \tag{4.2.68}$$

$$\chi^1_{Sijk}@n^3_k = \Pi^3_{Sij}, \quad (\sigma^1_{Sij} - T^1_{Sij})@N^3_j = F^3_{Si}, \quad on \quad \partial\Omega_S \cap \partial\Omega^{3+} \tag{4.2.69}$$

$$\chi^1_{Sijk}@(-n^3_k) = -\Pi^3_{Sij}, \quad (\sigma^1_{Sij} - T^1_{Sij})@(-N^3_j) = -F^3_{Si}, \quad on \quad \partial\Omega_S \cap \partial\Omega^{3-} \tag{4.2.70}$$

periodicity conditions on $\partial\Omega_F \cap \partial\Omega$,

$$\sigma^1_{Fij}@N^1_j = F^1_{Fi}, \quad on \quad \partial\Omega_F \cap \partial\Omega^{1+} \tag{4.2.71}$$

$$\sigma^1_{Fij}@(-N^1_j) = -F^1_{Fi}, \quad on \quad \partial\Omega_F \cap \partial\Omega^{1-} \tag{4.2.72}$$

$$\sigma^1_{Fij}@N^2_j = F^2_{Fi}, \quad on \quad \partial\Omega_F \cap \partial\Omega^{2+} \tag{4.2.73}$$

$$\sigma^1_{Fij}@(-N^2_j) = -F^2_{Fi}, \quad on \quad \partial\Omega_F \cap \partial\Omega^{2-} \tag{4.2.74}$$

$$\sigma^1_{Fij}@N^3_j = F^3_{Fi}, \quad on \quad \partial\Omega_F \cap \partial\Omega^{3+} \tag{4.2.75}$$

$$\sigma^1_{Fij}@(-N^3_j) = -F^3_{Fi}, \quad on \quad \partial\Omega_F \cap \partial\Omega^{3-} \tag{4.2.76}$$

In solid Ω_s, by averaging the balance equation (4.2.59) in volume Ω with respect to y and using the divergence theorem, we have

$$< \frac{\partial \sigma^0_{Sij}}{\partial x_j} > - < \frac{\partial T^0_{Sij}}{\partial x_j} > = - \mid \Omega \mid^{-1} \int_{\Omega_S} (\frac{\partial \sigma^1_{Sij}}{\partial y_j} - \frac{\partial T^1_{Sij}}{\partial y_j}) dV = - \mid \Omega \mid^{-1} \int_{\partial \Omega_S} (\sigma^1_{Sij} - T^1_{Sij}) N_j ds$$

$$= \quad - \mid \Omega \mid^{-1} \int_{\partial \Omega_S \cap \partial \Omega} (\sigma^1_{Sij} - T^1_{Sij}) N_j ds - \mid \Omega \mid^{-1} \int_{\Gamma} (\sigma^1_{Sij} - T^1_{Sij}) N_j ds \tag{4.2.77}$$

Using periodicity conditions on $\partial \Omega_S \cap \partial \Omega$, we get

$$< \frac{\partial \sigma^0_{Sij}}{\partial x_j} > - < \frac{\partial T^0_{Sij}}{\partial x_j} > = - \mid \Omega \mid^{-1} \int_{\Gamma} (\sigma^1_{Sij} - T^1_{Sij}) N_j ds \tag{4.2.78}$$

By averaging the balance equation (4.2.60) in volume Ω with respect to y and using the divergence theorem, we have

$$< T^0_{Sij} > = < \frac{\partial \chi^0_{Sijk}}{\partial x_k} > + \mid \Omega \mid^{-1} \int_{\Omega_S} \frac{\partial \chi^1_{Sijk}}{\partial y_k} dV = < \frac{\partial \chi^0_{Sijk}}{\partial x_k} > + \mid \Omega \mid^{-1} \int_{\partial \Omega_S} \chi^1_{Sijk} n_k ds$$

$$= < \frac{\partial \chi^0_{Sijk}}{\partial x_k} > + \mid \Omega \mid^{-1} \int_{\partial \Omega_S \cap \partial \Omega} \chi^1_{Sijk} n_k ds + \mid \Omega \mid^{-1} \int_{\Gamma} \chi^1_{Sijk} n_k ds \tag{4.2.79}$$

By using the same periodic conditions on $\partial \Omega_S \cap \partial \Omega$ and $\chi^1_{Sijk} n_k = 0$ on Γ and . We obtain

$$< T^0_{Sij} > = < \frac{\partial \chi^0_{Sijk}}{\partial x_k} > \tag{4.2.80}$$

In fluid Ω_F, by averaging the balance equation (4.2.61) in volume Ω with respect to y and using the divergence theorem, we get

$$< \frac{\partial \sigma^0_{Fij}}{\partial x_j} > = - \mid \Omega \mid^{-1} \int_{\Omega_F} \frac{\partial \sigma^1_{Fij}}{\partial y_j} dV = - \mid \Omega \mid^{-1} \int_{\partial \Omega_F} \sigma^1_{Fij} N_j ds$$

$$= - \mid \Omega \mid^{-1} \int_{\partial \Omega_F \cap \partial \Omega} \sigma^1_{Fij} N_j ds - \mid \Omega \mid^{-1} \int_{\Gamma} \sigma^1_{Fij} N_j ds \tag{4.2.81}$$

Using the same periodic conditions on $\partial \Omega_F \cap \partial \Omega$, we obtain

$$< \frac{\partial \sigma^0_{Fij}}{\partial x_j} > = - \mid \Omega \mid^{-1} \int_{\Gamma} \sigma^1_{Fij} N_j ds \tag{4.2.82}$$

(4.2.78)+(4.2.82) and using the interface condition equation (4.2.64), we obtain

$$< \frac{\partial \sigma^0_{Sij}}{\partial x_j} > - < \frac{\partial T^0_{Sij}}{\partial x_j} > + < \frac{\partial \sigma^0_{Fij}}{\partial x_j} > = 0 \tag{4.2.83}$$

Let us define the total stress σ^0 as

$$\sigma^0_{ij} = \begin{cases} \sigma^0_{Sij} & in\ \Omega_S \\ \sigma^0_{Fij} & in\ \Omega_F \end{cases} \tag{4.2.84}$$

Using the total stress and exchanging order of integration and derivative, we get

$$\frac{\partial < \sigma^0_{ij} >}{\partial x_j} - \frac{\partial < T^0_{Sij} >}{\partial x_j} = 0 \tag{4.2.85}$$

where

$$< \sigma^0_{ij} > = \mid \Omega \mid^{-1} \int_{\Omega_S} \sigma^0_{Sij} dV + \mid \Omega \mid^{-1} \int_{\Omega_F} \sigma^0_{Fij} dV = < a_{ijmn} > e_{xmn}(u^0_S) - \phi p^0 I_{ij} \tag{4.2.86}$$

A second compatibility condition is obtained from the fliud volume balance at order ε^0

$$\frac{\partial v^0_{Fi}}{\partial x_i} + \frac{\partial v^1_{Fi}}{\partial y_i} = 0 \tag{4.2.87}$$

By integrating over Ω with respect to y, and using the divergence theorem and the periodicity of v^1, we have

$$\begin{aligned} < \quad \frac{\partial v^0_{Fi}}{\partial x_i} > &= - \mid \Omega \mid^{-1} \int_{\Omega_F} \frac{\partial v^1_{Fi}}{\partial y_i} dv = - \mid \Omega \mid^{-1} \int_{\partial \Omega_F} v^1_{Fi} N_i ds \\ &= - \mid \Omega \mid^{-1} \int_{\partial \Omega_{F \cap \partial \Omega}} v^1_{Fi} N_i ds - \mid \Omega \mid^{-1} \int_{\Gamma} v^1_{Fi} N_i ds = - \mid \Omega \mid^{-1} \int_{\Gamma} v^1_{Fi} N_i ds \end{aligned} \tag{4.2.88}$$

By using the displacement continuity condition on Γ and $u^1_S = u^1_S(x,t)$, we get

$$< \frac{\partial v^0_{Fi}}{\partial x_i} > = - \mid \Omega \mid^{-1} \int_{\Gamma} v^1_{Fi} N_i ds = - \mid \Omega \mid^{-1} \int_{\Gamma} \frac{\partial u^1_{Si}}{\partial t} N_i ds = - \frac{\partial u^1_{Si}}{\partial t} \mid \Omega \mid^{-1} \int_{\Gamma} N_i ds \tag{4.2.89}$$

We look for Λ_i is a constant.

$$\frac{\partial \Lambda_i}{\partial y_i} = 0 \tag{4.2.90}$$

$$0 = \int_{\Omega_F} \frac{\partial \Lambda_i}{\partial y_i} dv = \int_{\partial \Omega_F} \Lambda_i N_i ds = \int_{\partial \Omega_F \cap \partial \Omega} \Lambda_i N_i ds + \int_{\Gamma} \Lambda_i N_i ds = \Lambda_i \int_{\Gamma} N_i ds \tag{4.2.91}$$

Hence, we have $\int_{\Gamma} N_i ds = 0$.

$$< \frac{\partial v_{Fi}^0}{\partial x_i} >= \frac{\partial < v_{Fi}^0 >}{\partial x_i} = 0 \tag{4.2.92}$$

We obtain macroscpic second grade modelling of saturated porous matrix by homogenization

$$\frac{\partial < \sigma_{ij}^0 >}{\partial x_j} - \frac{\partial < T_{Sij}^0 >}{\partial x_j} = 0 \tag{4.2.93}$$

$$< T_{Sij}^0 >= \frac{\partial < \chi_{Sijk}^0 >}{\partial x_k} \tag{4.2.94}$$

$$< \chi_{Sijk}^0 >=< b_{ijkhmn} + b_{ijkqrs} \frac{\partial}{\partial y_q} (e_{yrs}(\xi^{hmn})) > \frac{\partial}{\partial x_h} (e_{xmn}(u_S^0)) \tag{4.2.95}$$

$$< \sigma_{ij}^0 >=< a_{ijmn} > e_{xmn}(u_S^0) - \phi p^0 I_{ij} \tag{4.2.96}$$

$$\frac{\partial < v_{Fi}^0 >}{\partial x_i} = 0 \tag{4.2.97}$$

$$< v_{Fi}^0 > -\phi \frac{\partial u_{Si}^0}{\partial t} = \frac{1}{\mid \Omega \mid} \int_{\Omega_F} k_{ij} dV \frac{\partial p^0}{\partial x_j} \tag{4.2.98}$$

Chapter 5
Conclusions

This master project mainly focused on finding a macroscopic second grade model of a one dimensional periodic medium, an empty porous matrix and a saturated porous matrix using the asymptotic homogenization method. The results of this method present similarities with the original analysis performed with partial differential equations. The main difference that was observed in this work, is the fact that no boundary conditions applied in the sample.

Moreover, the double scale asymptotic expansion method was chosen as the appropriate method to study heterogeneous periodic materials and the model of homogenization method is equivalent model without the boundary condition of solid or interface condition between the solid and fluid.

By using the asymptotic homogenization method, second grade models could be determined for saturtated porous media.

Finally, the subject of this project has been studied to a certain extent, due to the short period that was dedicate for it. However, some perspectives for future work would include the application of these models to alternative, real materials. Also the numerical solution of the same models under loading would also be interesting to study future.

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